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Exact renormalization group study of fermionic theories*

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The exact renormalization group approach (ERG) is developed for the case of pure fermionic theories by deriving a Grassmann version of the ERG equation and applying it to the study of fixed point solutions and critical exponents of the two-dimensional chiral Gross-Neveu model. An approximation based on the derivative expansion and a further truncation in the number of fields is used. Two solutions are obtained analytically in the limit $N \to \infty$, with N being the number of fermionic species. For finite N some fixed point solutions, with their anomalous dimensions and critical exponents, are computed numerically. The issue of separation of physical results from the numerous spurious ones is discussed. We argue that one of the solutions we find can be identified with that of Dashen and Frishman, whereas the others seem to be new ones.

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1. Introduction

One of the major issues in QFT is the search for non-perturbative results. In particular little is known about the phase structure (fixed points, critical exponents, etc.) of most physically interesting theories. Nor have intriguing questions about the renormalization flows, like the proper extension of the c-theorem [1], been completely understood [2].

One of the methods capable of handling such problems is the exact renormalization group (ERG, hereafter). Originally developed by Wilson in his seminal articles in the early seventies [3] (see Ref. [4] for a classical review), it has recently attracted much attention. Although firstly used for studies of critical phenomena in condensed matter problems [5] its scope includes many other fields like particle theory, as it was demonstrated by Polchinski in an elegant paper where he proved the perturbative renormalizability of $\lambda\phi^4$ theory in a quite simple way [6]. Similar manipulations have led to quite interesting results regarding the study of different aspects of the perturbation expansion around a Gaussian fixed point and its associated diagramatic expansion [7,8]. However, the power of the ERG relies on the possibility of obtaining quantitative knowledge about the renormalization flows, in particular one piece of information which is probably the most valuable, due to its universality: the critical exponents [9].

The ERG approach is based on writing down a functional differential equation that expresses how the action changes when we integrate out high energy modes. This is the so-called ERG equation, and it is the cornerstone of the whole technique: with it and together with the most general action consistent with the symmetries of the model, the complete set of β-functions can be computed, and from these the location of the fixed points and their exponents. However, because of practical reasons it is impossible to handle all possible operators which could be included into the action. One must choose a more selective criterion, rather than simply to be consistent with the demanded symmetries, that is, one must choose a reasonable truncation of the general expansion. Usual approximations attempt to restrict the space of interactions to a reasonable number of operators, e.g. replacing the effective action for a non-derivative effective potential or expanding the action in powers of the momentum. For instance, to study the Wilson fixed point of $\lambda\phi^4$ in three dimensions, the authors of Ref. [9] consider only arbitrary polynomials in the fields without derivatives, but the same type of problem has also been addressed by changing the ERG equation and/or by considering other types of truncations [4,10-19].

ERG methods have been used in more complicated cases, like phase transitions at finite temperature [20], theories with gauge interactions [8,21] and theories with fermions as fundamental particles. In the last case, it usually relies on a certain kind of bosonization, usually consisting in coupling fermionic bilinears with scalar fields, and then studying self-interactions of these scalar fields[1] [22]. We feel, nonetheless, that this approach is unsatisfactory. The reason is that one should learn how to deal with pure spinor theories without simplifying the problem to a scalar one. Moreover, there are some quite interesting phenomenological models described entirely with spin 1/2 fields, like the celebrated Fermi theory of weak interactions [25], models for resonance physics [26] based on extensions of the Nambu-Jona-Lasinio action [27], or even models explaining

[1] The main exceptions are Ref. [23] which deals with perturbative properties of the Gaussian fixed point and the series of Ref.[24] which lead to the proof of the existence of the Gross-Neveu model in $2+\epsilon$ dimensions. We are seeking, however, non-perturbative quantitative information out of the ERG, rather than rigorous formal statements.

the symmetry-breaking sector of the electroweak theory, especially in connection with technicolor theories[2] [29]. It is also compelling the curiosity that all fields in the Standard Model can be expressible as fermionic fields [30]. Surely, further extensions have to be dealt with afterwards, like scalar and spin 1/2 fields coupled together and spinor particles interacting through gauge fields.

On the other hand, we remind the reader that fermions are not always easily manageable by non-perturbative methods, e.g. Lattice Field Theory. On the contrary, we will show that, once truncated, the ERG equation treats fermions and bosons very similarly, thus making possible a nearly immediate translation of knowledge from one case to the other.

Our purpose is to work with a sample theory based solely on spinor fields and to develop a method of obtaining numerical non-perturbative information from it (e.g. some critical exponents). With this motivation an ERG equation is derived, similar to the bosonic one by Polchinski [6], and applied to a particular model. We try to emphasize throughout the article that many of the peculiarities encountered are nothing more than the translation of their counterparts already found in previous papers on the ERG for bosonic theories.

To begin with, one may choose an appropriate model, relatively simple and non-trivial. As usual, the two-dimensional world is a perfect site where to look for. Indeed, the two-dimensional Clifford algebra is the simplest one, generated by the well-known Pauli matrices.

Moreover, the study of self-interacting fermionic theories in $d = 2$ can be traced back to the work of Thirring [31], where he proposed a massless model of a single Fermi field containing a quartic self-interaction. It can be solved exactly [32] and it presents some interesting features, most of them probably unexpected. It is perturbatively renormalizable, as power counting arguments suggest, but what is not expected from naive arguments is that it does not describe an interacting theory, but a trivial one [33].

The model becomes non-trivial when N species of fermions transforming under a global representation of the unitary group $U(N)$ are considered. This is the Gross-Neveu model [34], which is asymptotically free and renormalizable, within perturbation theory and also within the $1/N$ expansion. However, none of these approximations is capable to find any non-trivial fixed point for $d = 2$. (Actually the $1/N$ expansion shows a non-trivial fixed point of order $d - 2$ in d dimensions).

An interesting modification leads to the so-called chiral Gross-Neveu model [34], which is chosen to have the additional symmetry of the $U_R(N) \times U_L(N)$ chiral group. As in the previous case no fixed points, besides the Gaussian one, can be found within the $1/N$ approximation. Nevertheless, other non-perturbative techniques are available. Thus the quartic interaction of the chiral Gross-Neveu model can be expressed, after a Fierz transformation, as a current-current interaction and the latter allows an operator analysis of the model within the current algebra approach. Such a study is carried out in Ref. [35] where, exploiting conformal techniques, two critical curves in the space of couplings are found for which the theory is scale invariant. One of the lines corresponds to the abelian Thirring model, whereas the other one is truly non-trivial and does not pass through the origin. A very remarkable fact is that this result is exact and is not given by a zero of a β-function, neither in perturbation theory nor in the large N expansion. For this continuum set of critical theories the value of the coupling constant, associated to the abelian degrees of freedom, is arbitrary while the coupling associated to the $SU(N)$ degrees of freedom is fixed to be equal to

[2] Note, however, Ref. [28]

zero or $4\pi/(N+1)$. It is important to notice, however, that the fixed point is not unique and, as it happens in the Thirring model too, depends on an arbitrary parameter related to regularization ambiguities.

More recently, using bosonization, current algebra and conformal techniques, other non-trivial fixed points in two-dimensional fermionic models were found [36]. However a lot of work has to be done in order to gain a clearer understanding of the complete phase diagram.

Before ending this section let us describe shortly our results. We studied the chiral Gross-Neveu model with a Polchinski-type ERG equation and we projected the space of local interactions onto the 106-dimensional subspace generated by terms with at most six fermions and three derivatives.

Within the large N approximation we found two different non-trivial $N \to \infty$ limits, leading to qualitatively different results. One of them leads to a continuous family of fixed points along the direction of the $U(1)$ excitations, similar to that of Dashen and Frishman. However, unlike their case, the anomalous dimension η vanishes at leading order in $1/N$. The critical exponents can be computed analytically and most of them coincides with the canonical values. However for a wide class of regularization schemes the most relevant critical exponent is non-trivial and takes the value $\lambda_1 = 1.1231...$. The other solution gives a non-trivial (but scheme dependent) anomalous dimension ($\eta = 1.0 \sim 1.4$ in the range studied), but the fixed point is isolated. The relevant critical exponent in the same range is in this case $\lambda_1 = 2.18 \sim 2.26$.

For finite N the fixed point equations can be solved only numerically. We found a plethora of solutions, most of them spurious due to the nature of the approximation. As we will discuss later, it is very difficult to discriminate between good and fictitious solutions. We took specially care of those solutions that for large N match the $1/N$ fixed points. For both cases the solutions are strongly dependent on the scheme though the dependence can be mitigated using a minimum sensitivity criterion. In one case the solution is defined for any value of N, with $N\,\eta$ growing asymptotically to 4.9, and λ_1 decreases to its $1/N$ limit 1.123.... In the other case the solution disappears unexpectedly at $N \approx 142$ where it merges, as we will carefully describe, with another kind of solution. At the bifurcation point we have $\eta = 1.88$ and $\lambda_1 = 5.8$.

Finally we analyzed the case $N = 1$. This case has to be treated separately because in absence of flavour the Fierz transformations impose additional constraints that reduce considerably the total number of independent operators. The results in this case are disappointing: the fixed point solution is isolated and not a continuous family as in the Thirring model. This property is fulfilled in the previous order approximation (terms with less than three derivatives) and it is unclear for us why it is lost at this order. For this case the values of the anomalous dimension has an extremely wide variation with the renormalization scheme ($\eta = 1 \sim 10$).

Let us remind the reader that in general the fixed-point actions are scheme dependent and they contain any possible operator, thus making it very difficult to interpret them directly. Nevertheless, any fixed point is characterized by that piece of information that is universal and, thus, physically relevant. This contains the number of relevant directions and their associate exponents. For these reasons, we have refrained from presenting in detail the fixed-point actions but, rather, we have concentrated in discussing the universal properties of our results.

The article is organized as follows. In section 2 we derive an ERG equation for pure fermionic theories in any dimension. Sect. 3 is devoted to the construction of the action. The chiral Gross-Neveu model is defined through its symmetries and the truncation we use is explained. The calculation of the β-functions, fixed points and the corresponding critical exponents is divided into sections 4 and 5, while the sixth one

contains a summary of results and some conclusions. Because of the unavoidable increasing complexity of the notation (we will be dealing with 106 operators) we have included an Appendix (A) where all our conventions are summarized. Although any new symbol is clearly defined when it appears, we thought it would be helpful to have all them condensed in a single page. We have also included one appendix (B) to write down the whole action and another one (C) to present the complete set of β-functions.

2. ERG equation for fermionic theories

In this section we derive an ERG equation for a field theory written in terms of spinor quantities, on a Euclidean space of dimension d. Its role is to dictate the behaviour of the action as we integrate out modes, that is, how the action of our effective theory has to be modified when we vary the characteristic scale (cut-off) Λ, while keeping the S-matrix elements invariant. More concretely, if we parametrize the renormalization flow with $t \equiv -\ln \frac{\Lambda}{\Lambda_0}$, then the ERG equation will provide us with a sufficient condition to ensure $\dot{Z} = 0$, Λ_0 being a fixed scale and Z the generating functional for the connected Green's functions.

We will consider, thus, a general theory whose action is expressible as a function of spinor fields only[3] and artificially split it into

$$S = S_{kin} + S_{int} \ . \tag{2.1}$$

S_{int} is an arbitrary polynomial in the fields and momenta (we will work always in momentum space) and S_{kin} is a regulated version of the usual kinetic term,

$$S_{kin} = \int_p \overline{\psi}_{-p} P_\Lambda^{-1}(p)\, \psi_p \ , \tag{2.2}$$

where $\int_p$ stands for $\int d^d p$ and P_Λ is the matrix

$$P_\Lambda(p) = (2\pi)^d \frac{K_\Lambda(p^2)}{p^2} i\not{p} \ , \tag{2.3}$$

with $K_\Lambda(z)$ an analytic function over the whole finite complex plane that vanishes faster than any power-law when $z \to +\infty$ and is normalized to be $K_\Lambda(0) = 1$ [38].

In the following, we will consider expectation values defined by

$$\langle X \rangle \equiv \int \mathcal{D}\psi \mathcal{D}\overline{\psi}\ X\, e^{-S[\overline{\psi},\psi;\Lambda] + \int_p \overline{\chi}_{-p} Q_\Lambda^{-1}(p)\psi_p + \int_p \overline{\psi}_{-p} Q_\Lambda^{-1}(p)\chi_p + f_\Lambda} \ , \tag{2.4}$$

where X is any operator, χ_p, $\overline{\chi}_p$ are Grassmann sources, $Q_\Lambda(p)$ is another regulating function with analogous properties to $P_\Lambda(p)$ and, finally, f_Λ is a c-number independent of the fields.

With the above conventions, the starting point of the derivation is the observation that a functional integral of a total functional derivative vanishes, which leads us to

$$\begin{aligned} &\left\langle \left(\frac{\delta}{\delta\psi_p} - \overline{\psi}_{-p} P_\Lambda^{-1} + \overline{\chi}_{-p} Q_\Lambda^{-1} \right) \dot{P}_\Lambda \left(\frac{\delta}{\delta\overline{\psi}_{-p}} + P_\Lambda^{-1}\psi_p - Q_\Lambda^{-1}\chi_p \right) \right\rangle \\ &\qquad = -\left\langle \mathrm{tr}\left(P_\Lambda^{-1}\dot{P}_\Lambda \right) \delta(0) \right\rangle - \left\langle \left(\overline{\psi}_{-p} P_\Lambda^{-1} - \overline{\chi}_{-p} Q_\Lambda^{-1} \right) \dot{P}_\Lambda \left(P_\Lambda^{-1}\psi_p - Q_\Lambda^{-1}\chi_p \right) \right\rangle , \end{aligned} \tag{2.5}$$

[3] We assume that Weinberg's conjecture [37] is valid: an arbitrarily general action leads to arbitrarily general S-matrix elements and vice versa.

where the trace is over spinor indices. This is the counterpart of Eq. (1.9) of Ref. [38] and, as there, it can be used to identify the rate of change of the kinetic term,

$$\begin{aligned}\int_p \left\langle \overline{\psi}_{-p} \dot{P}_\Lambda^{-1} \psi_p \right\rangle &= \int_p \left\langle \frac{\delta S_{int}}{\delta \psi_p} \dot{P}_\Lambda \frac{\delta S_{int}}{\delta \overline{\psi}_{-p}} - \frac{\delta}{\delta \psi_p} \dot{P}_\Lambda \frac{\delta}{\delta \overline{\psi}_{-p}} S_{int} \right\rangle \\ + \int_p \left\langle \overline{\psi}_{-p} P_\Lambda^{-1} \dot{P}_\Lambda Q_\Lambda^{-1} \chi_p + \overline{\chi}_{-p} Q_\Lambda^{-1} \dot{P}_\Lambda P_\Lambda^{-1} \psi_p \right\rangle &+ \int_p \left\langle \overline{\chi}_{-p} Q_\Lambda^{-1} \dot{P}_\Lambda Q_\Lambda^{-1} \chi_p + \mathrm{tr} \left(P_\Lambda^{-1} \dot{P}_\Lambda \right) \delta(0) \right\rangle . \end{aligned} \tag{2.6}$$

On the other hand, by imposing that the generating functional is independent of the scale Λ we find the relation

$$\begin{aligned}\left\langle \dot{S}_{int} \right\rangle = &- \int_p \left\langle \eta\, \overline{\psi}_{-p} P_\Lambda^{-1} \psi_p + \overline{\psi}_{-p} \dot{P}_\Lambda^{-1} \psi_p \right\rangle - \frac{\eta}{2} \int_p \left\langle \overline{\psi}_p \frac{\delta S_{int}}{\delta \overline{\psi}_p} + \psi_p \frac{\delta S_{int}}{\delta \psi_p} \right\rangle \\ &+ \int_p \left\langle \frac{\eta}{2} \overline{\chi}_{-p} Q_\Lambda^{-1} \psi_p + \frac{\eta}{2} \overline{\psi}_{-p} Q_\Lambda^{-1} \chi_p + \overline{\chi}_{-p} \dot{Q}_\Lambda^{-1} \psi_p + \overline{\psi}_{-p} \dot{Q}_\Lambda^{-1} \chi_p \right\rangle + \left\langle \dot{f}_\Lambda \right\rangle , \end{aligned} \tag{2.7}$$

where the anomalous dimension is defined to be

$$\dot{\psi}_p = \frac{\eta}{2} \psi_p, \quad \dot{\overline{\psi}}_p = \frac{\eta}{2} \overline{\psi}_p. \tag{2.8}$$

We can now combine Eqs. (2.6) and (2.7) to write, after some straightforward algebra, an equation for $\left\langle \dot{S} \right\rangle$. It will be satisfied if

$$\begin{aligned}\dot{S} = &\int_p \left(\frac{\delta S}{\delta \psi_p} \dot{P}_\Lambda(p) \frac{\delta S}{\delta \overline{\psi}_{-p}} - \frac{\delta}{\delta \psi_p} \dot{P}_\Lambda(p) \frac{\delta S}{\delta \overline{\psi}_{-p}} \right) \\ &+ \int_p \left[\frac{\delta S}{\delta \psi_p} \left(\dot{P}_\Lambda(p) P_\Lambda^{-1}(p) \right) \psi_p - \overline{\psi}_{-p} \left(P_\Lambda^{-1}(p) \dot{P}_\Lambda(p) \right) \frac{\delta S}{\delta \overline{\psi}_{-p}} \right] . \end{aligned} \tag{2.9}$$

Note that our claim is that if (2.9) holds for functionals, as it stands, then a similar equation will hold for expectation values, which, in turn, ensures that all the Green's functions are invariant under the flow. We have just found, therefore, the sufficient condition we were looking for.

We still have to take into account the effects produced by the rescalings needed after any Kadanoff type of change in order to complete a RG transformation. We have partially included them when we consider not bare fields but renormalized ones with some anomalous dimension η. What is left over is just the canonical evolution of all quantities. To compute them in closed form the easiest way is to write (2.9) after having rescaled all variables with the appropriate powers of Λ to make them dimensionless,

$$\begin{aligned}\dot{S} = 2(2\pi)^d \int_p K'\left(p^2\right) \left(\frac{\delta S}{\delta \psi_p} i\not{p} \frac{\delta S}{\delta \overline{\psi}_{-p}} - \frac{\delta}{\delta \psi_p} i\not{p} \frac{\delta S}{\delta \overline{\psi}_{-p}} \right) \\ + dS + \int_p \left(\frac{1-d+\eta}{2} - 2p^2 \frac{K'\left(p^2\right)}{K\left(p^2\right)} \right) \left(\overline{\psi}_p \frac{\delta S}{\delta \overline{\psi}_p} + \psi_p \frac{\delta S}{\delta \psi_p} \right) - \int_p \left(\overline{\psi} p^\mu \frac{\partial'}{\partial p^\mu} \frac{\delta S}{\delta \overline{\psi}_p} + \psi_p p^\mu \frac{\partial'}{\partial p^\mu} \frac{\delta S}{\delta \psi_p} \right) , \end{aligned} \tag{2.10}$$

where the prime in $\partial'/\partial p^\mu$ means that the derivative does not act on the momentum conservation delta functions and thus only serves to count the powers of momenta of a given functional.

Note that, once the above ERG equation is derived, it would be easy to derive a similar one for a model involving Yukawa couplings just by combining the present manipulations with that of, for instance, Refs. [6,38]. (The resemblance of (2.10) and a Polchinski type equation for scalar theories is pretty evident.)

To fully specify the evolution of our theory under the RG flow, we have also to write down equations for the terms containing χ_p and $\overline{\chi}_p$ and the term with neither the sources nor the fields. Although we will not need them in our analysis, just for the sake of completeness we present the expressions obtained without further comments[4]:

$$Q_\Lambda(p) = P_\Lambda(p)\,\tilde{Q}\left(p^2\right) \;\;,\;\; f_\Lambda = -\int dt \int_p \tilde{Q}^{-2}\left(p^2\right)\overline{\chi}_{-p}\dot{P}_\Lambda^{-1}\left(p^2\right)\chi_p \;, \tag{2.11}$$

with $\tilde{Q}\left(p^2\right)$ a scalar function that evolves according to the equation $\dot{\tilde{Q}}\left(p^2\right) = \frac{\eta}{2}\tilde{Q}\left(p^2\right)$.

As a final comment about our equation is that we present it on Euclidean space as it is customary in the field. For our purposes, however, there is nothing special about the Euclidean formulation, as finally what one obtains is just a set of relations among coupling constants. In fact, we have also derived the counterpart of Eq. (2.10) for Minkowski space. It is not so nice because of the extra presence of an imaginary unit coming from the functional derivatives of the Minkowskian "Boltzmann" factor $e^{i\mathcal{S}}$ in the second term. Nevertheless, with this equation we have computed the β-functions for a simplified action (one without operators with six fields) in much the same way we will explain later for Euclidean space: they are finite, real and consistent with the desired symmetries, as they should be. We have not proceeded further, but the parallelism between them and their Euclidean counterparts strongly supports the common lore that both should contain the same physical information and that the choice of space is much a matter of taste. Nevertheless, it would probably be nice to afford a complete calculation in Minkowski space.

3. The action

In this section we begin the discussion of an explicit example. We first define it through its symmetries, then justify how one can truncate its general action while still retaining non-trivial information and, finally, we give the prescriptions we have actually used to build it systematically.

The sample model is that with N spin 1/2 two-dimensional Euclidean fields that obey the discrete symmetries of parity, charge conjugation and, to obtain reflection positive Green's functions, reflection hermiticity (see Ref. [39] for a precise definition of them). We further impose the continuous symmetries of Euclidean invariance and the chiral symmetry $U(N)_R \times U(N)_L$.

For the definition to be consistent, one has to check that the above classical symmetries of the action will survive after quantization. That is, one has to ensure that the symmetries will be satisfied at any point of the flow if they are satisfied by the initial conditions. In our case this is verified nearly immediately by just looking at Eq. (2.10). The point is that the Kadanoff terms, which are the eventually dangerous ones, essentially take the form, in spinor and flavour indices, of the free kinetic term of the action.

The next step is to choose an appropriate truncation. One would desire a kind of derivative expansion, at least because it is quite efficient when applied to bosonic theories [15,19]. However, the similarity with

[4] For a discussion with respect to the scalar case see again Ref. [38].

the scalar case cannot be carried that far. The first important difference is that, unlike the scalar case, the zero momentum approximation (effective potential) is not feasible and the leading order is one with zero and one derivative terms. The reason is almost evident: Eq. (2.10) contains, due to the sum over polarizations, a $\not{p}$ factor in the Kadanoff terms, while a similar equation for bosons does not[5].

Another significant difference is that in the scalar case a general potential contains an infinite number of independent functionals, whereas for finite N a general product of fermionic fields with fixed number of derivatives has in any case a finite number of terms due to the statistics. It is impossible to put twice the same Grassmann quantity at the same point. Thus, for the fermionic case the derivative expansion leads unavoidably to a polynomial approximation. This has practical consequences: the ERG equation becomes a large system of coupled non-linear ordinary differential equations instead of a small set of coupled partial differential equations, and the techniques to obtain numerical results are different. Furthermore, the number of different structures for an arbitrary large value of N grows extremely fast as the order of the derivatives increases. In practice, it becomes practically intractable at order 3 unless the degree of the polynomial of the fields is also truncated. For this reason we work, up to a finite number of derivatives and *also* up to a finite number of fields. The remaining decision is to choose where to truncate.

We require that a *sine qua non* property of a decent approximation is to allow a nontrivial anomalous dimension. Therefore, we will keep as many derivatives as needed to allow for a non-zero η, within a reasonable number of fields (of the order of, say, twice the number of derivatives). With this criterion it is easy to realize that one derivative and four fields do not work: only the Gaussian fixed point is obtained, with classical critical exponents. Two derivatives seem in principle sufficient. However, once the β-functions are obtained, it can be shown that the result $\eta = 0$ is unavoidable, thus forcing us to work with terms up to three derivatives and six fields.

The final preparatory step is to write down the action. To construct it systematically we list all symmetries and study the restrictions imposed by each of them.

We will work with the momentum representation and, in order to simplify the notation, we will take the convention that any product of fields should be eventually integrated over the momentum carried by each field, with a common momentum conservation delta function. This would correspond to an integral over the whole space of a product of fields and their derivatives (of any order) at the same point.

i) $\underline{U(1)}$. We begin with $U(1)$, fermion number conservation. Its consequences are well known: the action must be built up of operators of the form

$$S_{12}^{ab} \equiv \overline{\psi}^{a}\left(p_{1}\right) \psi^{b}\left(p_{2}\right), \quad P_{12}^{ab} \equiv \overline{\psi}^{a}\left(p_{1}\right) \gamma_{s} \psi^{b}\left(p_{2}\right), \quad V_{12}^{j,ab} \equiv \overline{\psi}^{a}\left(p_{1}\right) \gamma^{j} \psi^{b}\left(p_{2}\right), \tag{3.1}$$

where we work in the momentum representation, a, b denote flavour indices and from now on the subindices of S, P, V label the fermion momenta. The Clifford algebra is defined by $\left\{\gamma^{i}, \gamma^{j}\right\} = 2\delta^{ij}$ with $\gamma_s = -i\gamma^1\gamma^2$. Note that in two dimensions there are no other spinorial structures, since $\gamma_s \gamma^j = i\epsilon^{jk}\gamma^k$.

ii) *Euclidean invariance*. The Euclidean invariance is also easily taken into account: one has only to make sure that all Euclidean indices are properly contracted.

iii) $\underline{SU(N)}$. The next one is the (vector) $SU(N)$ group. If the fields transform under the fundamental representation, all possible scalar operators can be classified with the aid of Fierz reorderings. In fact, it is not

[5] See, for instance, Eq. (18) of Ref. [6].

difficult to show by means of Fierz transformations that a general local operator in the trivial representation, built from products of fermionic fields, can be factored in terms of

$$S_{12} \equiv S_{12}^{aa}, \quad P_{12} \equiv P_{12}^{aa}, \quad V_{12}^{j} \equiv V_{12}^{j,aa}. \tag{3.2}$$

Thus, the simplest manner to get rid of the internal group indices is to work with a basis written as products of scalar, pseudoscalar and vector operators (S, P, V^j), transforming under the trivial representation, and powers of momenta. Therefore, the simplicity of two dimensions has come to help us again: a general functional can be written in terms of only three "building blocks", and momenta.

iv) $\underline{SU_R(N) \times SU_L(N)}$. To enlarge $SU(N)$ to $SU_R(N) \times SU_L(N)$ we realize that the chiral invariant operators are constructed from the combinations

$$V_{12}^{j}, \quad S_{12}S_{34} - P_{12}P_{34}, \quad S_{12}P_{34} - P_{12}S_{34}. \tag{3.3}$$

Therefore, if we restrict our attention to those kinds of terms, again with an arbitrary structure of momenta, the symmetry will be fulfilled.

Note that the first type of operator in (3.3) carries a space index j and two fields, whereas the other two have no indices and four fields. From this it can be immediately inferred that with an even number of derivatives one can only have operators with $4n$ fields (n integer), whereas with an odd number of derivatives the allowed operators contain $4n+2$ fields. The reason is that all indices must be contracted, either by the Kronecker delta δ^{ij} or by the complete antisymmetric tensor in two dimensions ϵ^{ij}, which implies that an odd number of derivatives needs an odd number of operators of the type V_{12}^{j}.

v) *Parity*. It only remains to impose discrete symmetries. Parity is easy: for a Euclidean invariant operator, products of S, V^j and momenta are parity-conserving. The only problem is when we have the pseudo-scalar operator P. What we have to do is just follow the standard rule: a term with an odd number of P's must contain a Levi-Civita symbol ϵ^{ij} also; a term with an even number of P's must not.

vi) *Charge conjugation*. To impose charge conjugation and reflection hermiticity proves to be the most involved task. This is because both operations exchange fermions and antifermions, and thus they change, in general, the momentum structure. Explicitly, under charge conjugation our elementary operators transform as

$$S_{12} \to S_{21}, \quad P_{12} \to -P_{21}, \quad V_{12}^{j} \to -V_{21}^{j}. \tag{3.4}$$

To take into account this symmetry at the level of the basis, the most effective manner is to consider all momenta written in combinations like $(p_1 \pm p_2)^j$, where p_1 is the momentum of an antifermion and p_2 the momentum of the fermion of the same bilinear. In this way it is easy to distinguish between C-conserving and C-violating operators, and to construct both sets.

vii) *Reflection hermiticity*. The last one is reflection hermiticity. It is defined, in principle, in coordinate space [39] and under such transformation, our "elementary operators" behave just as in (3.4). What is new is that when transformed, one must change the coefficient of the operator by its complex conjugate. Therefore, once we restrict ourselves to C-conserving terms, this additional symmetry restricts the coefficients of those terms to be real. The only subtlety is that, as it is defined, the fields do not become complex

conjugated and neither do their derivatives[6]. And if one remembers that a derivative in coordinate space amounts to a factor $-ip$ this indicates that an extra power of i should be added for each power of momentum.

viii) *Further degeneracies*. Finally, the freedom of integrating by parts (each operator has a delta function of momenta conservation) relates different functionals, and, ultimately, reduce the number of independent ones. The best way of implementing these final constraints is to find out a criterion in order to write down every operator in a "standard" way. We will explain ours in Appendix A, where we will also write down the complete action, consisting of a basis of 107 functionals.

One of them is rather peculiar. It is

$$iV_{12}^{j}(p_1 - p_2)^j (p_1 - p_2)^2. \tag{3.5}$$

One may be worried about it because it would lead to a propagator with an additional pole besides the physical one on the particle mass-shell, thus entering in conflict with unitarity. This is, however, not important at all, because the above kind of reasoning implies that one assumes a well-defined perturbative expansion, and this is not the case (we have irrelevant operators that make any perturbative expansion around the Gaussian fixed point completely ill-defined). One should think that the theory is such that it manages to have a well-defined complete two-point function free from unphysical singularities. A completely different point is that, besides the above discussion, when one computes the β-functions of the theory one realizes that this operator, at least up to the order we are considering, does not contribute to any other. Therefore its evolution will affect absolutely no conclusion we obtain without it. For this reason, we do not include it in the action. We should remark, however, that for all we do it is as if this term were already there, although for the sake of brevity we will not write it down any more.

4. Computing the β-functions

Once we have constructed the initial action we want to work with, the next step is to substitute it into the ERG equation (2.10) and to compute the β-functions of our model within the given approximation.

In principle this is just a purely algebraic exercise. Nevertheless, it turns out that from a practical point of view it is an almost forbidding task, if done by hand. During intermediate steps of the calculation one has to handle thousands of terms and it is too easy to make errors. For instance, when computing $\frac{\delta S}{\delta \psi_p} \dot{P}_\Lambda (p) \frac{\delta S}{\delta \overline{\psi}_{-p}}$, the functional differentiation gives 302 terms, and one has, roughly speaking, to square them and multiply the result by the inverse propagator. Then, one has to compute the appropriate products of gamma matrices, expand all the terms and, finally, perform the integration by parts to reach our chosen basis. The number of operators considerably increases in these last processes. Thus, it is mandatory to use a symbolic manipulation computer program to perform the functional differentiation, do the algebra and integrate by parts. Because of this, our computation was done with the help of *Mathematica*.

To calculate the flow equations, we use an extended action, greater than that discussed so far, in order to have some extra check of our equations. That is, we consider an action expanded in a basis that consists

[6] We remind the reader that, in order to turn properly from Minkowski to Euclidean space, one has to redefine the symmetries of the problem, specially those which involve complex conjugation. Our definitions coincide with those of Ref. [39].

of terms with two fermions with one and three derivatives, four fermions with zero and two derivatives and six fermions with one and three derivatives, but without imposing any symmetry other than vector $U(N)$, parity and Euclidean invariance (that is, we impose neither reflection hermiticity, nor charge conjugation, nor chirality). We then project the space generated by this basis into the invariant subspace under the required symmetries and its direct complement. The required flow equations are obtained after the first projection, while the complementary subspace provide us a consistency check of the calculation. They define a set of null equations that have to be satisfied along the renormalization flow: after projecting to an initial symmetric action, any non-zero contribution of a non-symmetric term will indicate an anomaly, which we have argued are non-existing. We leave to Appendix B the complete set of β-functions.

Finally let us justify the inclusion of operators with three derivatives into the action. As we advanced above, it is motivated by the fact that those terms are necessary in order to get a non-vanishing critical anomalous dimension. The argument is as follows. The anomalous dimension is related to the fact that we are free to fix the normalization of one term of the action, by choosing an appropriate normalization of our fields. If, as is customary, we keep fixed the coefficient of the so-called kinetic term, then its β-function is substituted by an equation for η which, in practice, is calculated in a similar fashion. We have to study, then, which type of Kadanoff transformations contribute to $\overline{\psi}_{-p}\not{p}\psi_p$. The term $\frac{\delta S}{\delta\psi_p}i\not{p}\frac{\delta S}{\delta\overline{\psi}_{-p}}$ cannot, unless there were a mass operator, which is forbidden by chiral symmetry. There are, however, some contributions coming from $\mathcal{S}^{(4,2)}$, due to $\frac{\delta}{\delta\psi_p}i\not{p}\frac{\delta S}{\delta\overline{\psi}_{-p}}$. We will find, hence, the anomalous dimension as a linear combination of couplings of $\mathcal{S}^{(4,2)}$ and, consequently, if these couplings vanish at the fixed point then $\eta = 0$ is unavoidable. (We define $\mathcal{S}^{(a,b)}$ as the part of the action that contains a fields and b derivatives). If one now studies $\dot{\mathcal{S}}^{(4,2)}$, it is not difficult to convince oneself that its only contributions must come from $\mathcal{S}^{(6,3)}$, apart from canonical rescalings. The implications are now immediate: if $\mathcal{S}^{(6,3)}$ did not exist, then the whole action $\mathcal{S}^{(4,2)}$ would evolve canonically, thus it would vanish at the fixed point and we would obtain a vanishing anomalous dimension.

5. Fixed points, critical exponents

5.1. Generalities

The next step is to find the fixed point solutions, that is, the sets of coupling constants that make all the β-functions vanish. These will indicate the points to which the RG tends to, thus providing us with the first indication of what the phase diagram of the system looks like.

The condition $\dot{S} = 0$ is equivalent to a system of 106 non-linear algebraic equations. To simplify it we note that all the coupling constants of operators with six fields must enter linearly, because the only source of non-linearity of Eq. (2.10) is its first term on the r.h.s., and it can give contributions neither from $\mathcal{S}^{(6,1)}$ nor from $\mathcal{S}^{(6,3)}$, within our approximation. Therefore, we can reduce the system to a set of only 13 non-linear equations,

$$\begin{aligned}
0 &= 2\eta g_1 + 8g_1^2\alpha\gamma N/(-2+3\eta) + \{8g_1^2\beta\delta N + 8\beta\gamma[g_1(-4r_2 + s_1 - s_2 - 3s_3 - 3s_4) \\
&\quad + 2g_2(m_1 - m_2 - m_3) + 4g_1 m_3 N]\}/(-4+3\eta), \\
0 &= 2\eta g_2 + 8g_1^2\alpha\gamma/(-2+3\eta) + \{8g_1^2\beta\delta + 8\beta\gamma[2g_1(m_1 - m_2 + m_3 - s_2)
\end{aligned}$$

$$
\begin{aligned}
&+ g_2(-4r_2 + s_1 - s_2 - 3s_3 - 3s_4) + 2g_2 s_2 N]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)m_1 + \{16 g_1 \alpha\delta(-2g_2 + g_1 N) + 4\alpha\gamma[g_1(m_1 - m_2 - m_3 - 2r_1 - 6r_3 - 4s_2 - 3s_3 + s_4 - t) \\
&\quad - g_2(3m_1 + m_2 + 5m_3 + 2t) + 2g_1 N(2m_1 + 3m_3 + r_2 + 2s_3 + s_4 + t)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)m_2 + \{16 g_1 g_2 \alpha\delta + 2\alpha\gamma[g_1(2r_1 + 2r_2 + 6r_3 - 3s_1 + 5s_2 + s_3 + 5s_4) \\
&\quad + 2g_2(m_1 + m_2 + 3m_3 - t)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)m_3 + \{16 g_1 g_2 \alpha\delta + 2\alpha\gamma[g_1(2m_1 - 2m_2 - 2m_3 + 2r_1 - 2r_2 + 6r_3 - s_1 - s_2 - 7s_3 - 3s_4 - 2t) \\
&\quad + 2g_2(2m_3 + 2m_1 - t) + 4g_1 N(m_3 + r_2 + 2s_3 + s_4)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)r_1 + \{8 g_2 \alpha\delta(2g_1 - g_2 N) + 4\alpha\gamma[g_1(m_1 - m_2 + m_3 + 4r_3 - 2s_2) \\
&\quad + g_2(3m_1 + m_2 + m_3 - 2r_2 + s_1 - 3s_2 - 4s_3 - 4s_4 - t) + 2g_2 N(r_2 - 2r_3 + s_2 + 2s_3 + s_4)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)r_2 + \{4(g_1^2 + 4g_2^2)\alpha\delta + 4\alpha\gamma[g_1(2m_3 - t) + g_2(2r_1 + 6r_3 - s_1 + 3s_2)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)r_3 + \{12 g_1^2 \alpha\delta + 4\alpha\gamma[g_1(3m_1 + m_2 + 5m_3 - t) \\
&\quad + g_2(m_1 - m_2 - m_3 - s_1 - s_2 - 3s_3 + s_4 - t) + 2g_2 N(r_2 + 2s_3 + s_4)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)s_1 + \{16 g_2 \alpha\delta(2g_1 - g_2 N) + 16\alpha\gamma[g_1(2r_3 + s_2) + g_2(m_1 + m_2 + m_3) \\
&\quad - g_2 N(2r_3 + s_2)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)s_2 - \{8 g_1^2 \alpha\delta + 8 g_1 \alpha\gamma(m_1 + m_2 + m_3)\}/(-4+3\eta), \\
&0 = 2(-1+\eta)s_3 + \{4 g_1^2 \alpha\delta + 4\alpha\gamma[g_1(2m_1 + 2m_2 + t) + g_2(-2r_1 + 2r_3 + s_1 + s_2)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)s_4 + \{4 g_1^2 \alpha\delta + 4\alpha\gamma[g_1(2m_3 - t) + g_2(2r_1 - 2r_3 - s_1 - s_2)]\}/(-4+3\eta), \\
&0 = 2(-1+\eta)t + 8\alpha\gamma[g_1(-2r_1 + 2r_3 + s_1 + s_2) + g_2(m_1 + 3m_2 - m_3) + g_1 N(-2m_2 + t)]/(-4+3\eta), \quad (5.1)
\end{aligned}
$$

where η is the anomalous dimension that turns out to be

$$\eta = 4\alpha[-m_1 + m_2 + m_3 + s_1 + s_2 + s_3 + s_4 + t - 2N(r_2 + 2s_3 + s_4)]. \tag{5.2}$$

N is the number of flavours and α, β, γ, δ are scheme dependent parameters defined as

$$\alpha = \frac{1}{(2\pi)^d}\int_p K'\left(p^2\right), \quad \beta = \frac{1}{(2\pi)^d}\int_p p^2 K'\left(p^2\right), \quad \gamma = K'(0), \quad \delta = K''(0). \tag{5.3}$$

The appearance of the above quantities just reflects the freedom in choosing a renormalization scheme. Furthermore, although the β-functions depend on four parameters, we will see that the fixed point solution will depend only on two combinations of them. This is just the pattern that occurs in a scalar theory within a similar truncation [19].

The system cannot be solved analytically, unless we perform further simplifications, like keeping only the dominant term in an asymptotic expansion at $N \to \infty$. On the other hand, one can, of course, simply try to study its solution numerically. We will present both approaches in turn.

After the fixed points are identified, the behaviour of the theory near each of them is controlled by the critical exponents. One of them is fixed once we solve our set of equations: it is the anomalous dimension at the fixed point value. The rest are found by linearizing the RG transformations near the chosen fixed point. That is, if g_i is a generic coupling constant, then its variation in the vicinity of a fixed point g_0 is approximated by $\delta\dot{g}_i = \dot{g}_i = R_{ij}|_{g_0}\delta g_j$, where R_{ij} is the matrix $\frac{\partial \dot{g}_i}{\partial g_j}$. The eigenvalues of $R_{ij}|_{g_0}$ can be

identified with critical exponents. They can be thought of as the anomalous dimension of the operators which drive the theory away from the fixed point.

We can now no longer work with the reduced system of 13 couplings, but the full 105×105 matrix is needed as we allow deviations from the fixed point values of the six-fermion couplings. Around the Gaussian fixed point, for instance, the four-fermion and two-derivative operators have the same degree of "irrelevance" as the six-fermion and one-derivative ones.

Finally, let us turn again to the issue of scheme dependence. We have just said that, in general, the precise values of the coupling constants at a fixed point are scheme dependent, thus reflecting that they are not universal quantities. Critical exponents, on the other hand, are universal, hence they should be scheme independent. Nonetheless, due to the truncation, scheme dependencies will inevitably appear. What we will do is, as usual, to try to find a suitable scheme where the dependence will not be that important. To this end, we will apply to the various solutions a principle of minimal sensitivity to discriminate among different results [40,19].

5.2. $N \to \infty$

We are now going to set up a large N expansion for our model, with which analytic results can be obtained. Later on we will see that when we study the general case by suitable numerical approximations, we will recover our present results as a first term of the asymptotic series around $N \to \infty$.

To define properly our approximation, we substitute each coupling constant g_i by $N^{z_i} g_i$ and study the limit $N \to \infty$ keeping g_i fixed. In principle, z_i can be any real number, but for simplicity we only consider integer values. We then find the set $\{z_i\}$ that makes all β-functions finite and, if possible, non-trivial.

With the above requirements, there are essentially two different manners to define the $1/N$ expansion, which lead to different results. We label them by I and II, and discuss each in turn.

The Type I solution is obtained by considering $z_i = -1$, where i runs over every of the couplings that enter in Eq. (5.1). With this definition, the anomalous dimension vanishes at leading order in $1/N$ and the system (5.1) becomes

$$\begin{aligned}
&0 = -4\alpha\gamma g_1^2 - 2\beta\delta g_1^2 - 8\beta\gamma g_1 m_3 = -4g_2 s_2 \beta\gamma, \\
&0 = -2m_1 - 4\alpha\delta g_1^2 - 2\alpha\gamma g_1(2m_1 + 3m_3 + r_2 + 2s_3 + s_4 + t) = -2m_2 = -2m_3 - 2\alpha\gamma g_1(m_3 + r_2 + 2s_3 + s_4), \\
&0 = -2r_1 + 2\alpha\delta g_2^2 + 2\alpha\gamma g_2(-r_2 + 2r_3 - s_2 - 2s_3 - s_4) = -2r_2 = -2r_3 - 2\alpha\gamma g_2(r_2 + 2s_3 + s_4), \\
&0 = -2s_1 + 4\alpha\delta g_2^2 + 4\alpha\gamma g_2(2r_3 + s_2) = -2s_2 = -2s_3 = -2s_4, \\
&0 = -2t + 2\alpha\gamma g_1(2m_2 - t).
\end{aligned} \tag{5.4}$$

Its solution is

$$\begin{aligned}
&g_1 = -1/(\alpha\gamma), \quad m_2 = 0, \quad m_3 = \delta/(4\alpha\gamma^2) + 1/(2\beta\gamma), \quad r_1 = \alpha\delta g_2^2, \quad r_2 = r_3 = 0, \\
&s_1 = 2\alpha\delta g_2^2, \quad s_2 = s_3 = s_4 = 0, \quad t = 5\delta/(4\alpha\gamma^2) - 6/(4\beta\gamma) - m_1.
\end{aligned} \tag{5.5}$$

We now choose $z_i = -2$ for all the six fermions coupling constants. This is not the only solution since there are other rescalings consistent with the reduced system (5.4). For example, one can assign the value

-1 to some of the z_i's and -2 to the other ones, but it turns out that the results below do not depend on that.

The characteristic polynomial $P(\lambda)$ associated to the matrix of linear deviations is exactly computable,

$$\begin{aligned} P(\lambda) =& \lambda^2\ (\lambda+2)^{12}\ (\lambda+4)^{83}\ (\lambda+6)\ (\lambda^2+6\lambda-8) \\ & \times \left(-\lambda^5-12\lambda^4+(8w-44)\lambda^3+(64w-16)\lambda^2+(32w+64)\lambda-(128w+256)\right), \end{aligned} \tag{5.6}$$

where $w=\beta\delta/(\alpha\gamma)$ and $z=\delta/\gamma^2$. We can read the critical exponents from $P(\lambda)$. There are 100 scheme independent eigenvalues, most of them coinciding with the Gaussian values $0,-2,-4$ and -6. The non-trivial ones are $-3+\sqrt{17}=1.1231...$ and $-3-\sqrt{17}=-7.1231...$, and the roots of the polynomial

$$Q(\lambda)=-\lambda^5-12\lambda^4+(8w-44)\lambda^3+(64w-16)\lambda^2+(32w+64)\lambda-(128w+256), \tag{5.7}$$

which are all w-dependent. If $w<0$, that corresponds, for instance, to the exponential cut-off function $K_\Lambda(p^2)=e^{-\kappa\frac{p^2}{\Lambda^2}}$, the more relevant critical exponent is $\lambda_1=1.1231...$

Note that the fixed point solution depends freely on g_2 and m_1. This is the expected result for the chiral Gross-Neveu model because the $U(1)$ Thirring like excitations (which in our action are controlled by g_2) decouple from the rest and this subsystem is conformally invariant (i.e. it is at fixed point) for any value of g_2. For the $SU(N)$ part there exists a discrete set of fixed points, the one of Dashen and Frishman being one of them. This critical point is reached when the constant g_1 is of order $1/N$, as in our case. So we can make a correspondence between our solution and that of Ref. [35]. Nevertheless, the values of the anomalous dimension in both cases do not match. For the cited fixed point it is non-vanishing at leading order in $1/N$, and not zero as we have found. This discrepancy with the exact result of Dashen and Frishman could be caused by our truncation. We cannot reject, however, the possibility of having found a different fixed point as it has already occured previously [36].

For the Type II solution it is useful to define the new variable $m_1'=m_1-m_2+m_3$ instead of m_1. Then it corresponds to the following rescaling of couplings

$$\begin{aligned} & g_1\to g_1/N,\quad g_2\to g_2/N,\quad m_1'\to m_1',\quad m_2\to m_2/N,\quad m_3\to m_3/N,\quad r_1\to r_1/N,\quad r_2\to r_2/N,\quad r_3\to r_3/N, \\ & s_1\to s_1/N,\quad s_2\to s_2/N,\quad s_3\to s_3/N,\quad s_4\to s_4/N,\quad t\to t. \end{aligned} \tag{5.8}$$

The solution takes the form

$$\begin{aligned} g_1 =& \beta\frac{2\alpha-\beta\gamma}{48\alpha^3},\quad g_2=\beta\frac{-2\alpha+\beta\gamma}{24\alpha^3}, \\ m_1' =& \frac{-8\alpha+\beta\gamma}{144\alpha^2},\quad m_2=\frac{8\alpha-\beta\gamma}{72\alpha^2},\quad m_3=\frac{24\alpha^2\beta\delta+32\alpha^3\gamma-18\alpha\beta^2\delta\gamma-18\alpha^2\beta\gamma^2+3\beta^3\delta\gamma^2+4\alpha\beta^2\gamma^3}{576\alpha^3\gamma(-4\alpha+\beta\gamma)}, \\ r_1 =& \frac{43}{36\alpha}+\frac{\beta^2\delta}{12\alpha^3}-\frac{\beta\delta}{6\alpha^2\gamma}-\frac{43\beta\gamma}{288\alpha^2},\quad r_2=\frac{-8\alpha+\beta\gamma}{288\alpha^2},\quad r_3=\frac{8\alpha-\beta\gamma}{36\alpha^2}, \\ s_1 =& \frac{20}{9\alpha}+\frac{\beta^2\delta}{6\alpha^3}-\frac{\beta\delta}{3\alpha^2\gamma}-\frac{5\beta\gamma}{18\alpha^2},\quad s_2=\frac{8\alpha-\beta\gamma}{144\alpha^2},\quad s_3=\frac{-8\alpha+\beta\gamma}{288\alpha^2},\quad s_4=\frac{-8\alpha+\beta\gamma}{288\alpha^2}, \\ t =& \frac{8\alpha-\beta\gamma}{144\alpha^2}, \end{aligned} \tag{5.9}$$

which has a non-zero anomalous dimension,

$$\eta=\frac{4}{3}-\frac{\beta\gamma}{6\alpha}. \tag{5.10}$$

	$z = 0.1$	$z = 0.5$	$z = 1.0$	$z = 2.0$
λ_1^*	2.258	2.239	2.217	2.175
w^*	0.122	0.616	1.250	2.610
η	1.130	1.128	1.125	1.116

Table I: Local minimum of λ_1, the most relevant critical exponent, corresponding to the solution for $N \to \infty$ labelled as Type II in the text, for different values of z. η is the anomalous dimension at that point and w^* is the value of the parameter w at which the minimum is reached.

The set of the remaining z_i's is unique and composed of the numbers -1 and -2. Unfortunately, unlike the previous case we could not find the exact analytical expression of the characteristic polynomial. However by computing numerically the eigenvalues for different values of z and w we could guess some exact results. None of the critical exponents coincide with their canonical counterparts. Moreover, most of them are functions of the combination $\frac{w}{z}$. Thus there are 82 eigenvalues $\lambda = -\frac{w}{2z}$, 8 eigenvalues equal to $\frac{2}{3}(1 - \frac{w}{2z})$ and 4 of the form $2 - \frac{w}{2z}$. The remaining ones are not functions of the ratio w/z only (and even a few have a non-vanishing imaginary part, which is not unusual in approximations based on truncations). We have to study numerically the most relevant critical exponent, which belongs to the class with no simple dependence in w and z, for different scheme parametrizations. As it happens in the scalar case, for any value of z, this exponent always has a minimum at some scheme parametrized by $w = w^*$. This behaviour induced us to use the minimal sensitivity criterion to fix the parameter w to its critical value w^*. Unfortunately due to the monotonic dependence of the solution on the parameter z in the range analyzed, we were unable to set it with a similar prescription. In Table I we show some values of $\lambda_1^* = \lambda_1(w^*)$ and the anomalous dimension for various z's.

5.3. Finite N

For a finite number of flavours analytical results for the fixed point couplings cannot be found. So one has to proceed numerically to search for the zeroes of the β-functions. Moreover, the number of different solutions of a system of coupled non-linear equations is not known a priori and the common routines for root-finding (such as the FindRoot command of *Mathematica*) do not guarantee that all the zeroes are reached. A more serious inconvenience is to decide if a given zero corresponds to a real fixed point solution or if it is a spurious root resulting from our truncation.

The first problem can be reasonably reduced after some experience is acquired. In fact, we can know by intuition which is a reasonable range of values for the couplings and inspect this region exhaustively. Of course this is not easy for a system of thirteen equations, but we can gain some confidence in the results if we examine minutely the adequate region.

The second problem, however, is much more complicated. In principle we do not have any criteria to decide if a root of the β-functions system corresponds to a genuine fixed point solution or if it is a fictitious artifact of our approximations. This problem, which already appeared in the bosonic case too, is perhaps the Achilles' heel of the approximations based on truncations [17,18]. We present the class of solutions of

which we are more confident. These are mainly the ones which asymptotically match with some solution clearly identified in the framework of the large N expansion.

Let us, to begin with, select a particular scheme and find the solution for different values of N: $w = -2$ and $z = 0.5$, corresponding to an exponential regulating function ($K(x) = e^{-x^2}$). We analyze the dependence on these parameters later on.

One solution is found which behaves asymptotically as the type I one in the $N \to \infty$ limit. $N\eta$ increases with N and tends to 4.87..., while the most relevant critical exponent λ_1 decreases with N asymptotically to the value 1.1231..., in agreement with the $1/N$ expansion. For the second eigenvalue we find complex figures that we attribute to our approximations. Another piece of bad news is that, unlike the $N \to \infty$ case, the solution for finite though big enough N, is isolated, while, as we mentioned before, the fixed point solutions for Thirring like models are continuous in the $U(1)$ sector. Again we blame this confusing result on the truncation. We present in Fig. 1 the curves for $N\eta$ and λ_1 as a function of N.

More interesting is perhaps the study of the dependence of the solution on the scheme. We have noticed that z enters the equations only through the anomalous dimension as a global factor. For this reason, the dependence of the fixed points solutions in z is quite simple: it is almost linear in η. Therefore it is more attractive to investigate its behaviour under changes on the parameter w for fixed N and z. The motivation of this analysis is the search, as in the scalar case, of some non-linear w-dependence in such a way that we can invoke a principle of minimum sensitivity to fix the value of this parameter and eliminate one fictitious dependence. To this end, we fix the value of z to $z = 0.5$ and N to $N = 1000$. The curve η vs. w is monotonic and decreases with w, while the first eigenvalue λ_1 reaches its minimum value $\lambda_1 = 1.12511$ at $w = -45$, which increases as we lower N: it is equal to 1.1273 for $N = 500$ (it is reached at $w = -23$), 1.146 for $N = 200$ (at $w = -10$), 1.1519 for $N = 100$ (at $w = -8$), 1.695 for $N = 10$ (at $w = -2.4$) and finally, 2.560 for $N = 3$ (at $w = -0.5$). We show two of these curves in Fig. 2.

For the fixed point that matches the Type II solution as $N \to \infty$ we found a curious behaviour. For N moderately large, (say $N = 1000$), the numerical solution is in good agreement with the $1/N$ analytical result (for example, the value of η for $z = 0.5$ and $w = -2$ is 1.99, compared with the exact $\eta = 2$ for $N \to \infty$). As we lower N the values of the anomalous dimension η and the most relevant eigenvalue λ_1 decrease. But, unexpectedly, the solution disappears at $N = 142$ (actually at $N = 142.8$ if we let N take non-integer values). A closer analysis of the space of solutions shows us that at this value of N the branch of solutions compatible with the type II $1/N$ expansion merges with another family of fixed points. This last branch has finite asymptotic limits for η and λ_1 as $N \to \infty$. However some couplings do not behave as a power of N in this limit and, therefore, it cannot be associated with a $1/N$ fixed point in the sense stated previously. At the bifurcation point $\eta = 1.88$ and $\lambda_1 = 5.80$. We show in Fig. 3 the curves $\eta(N)$ and $\lambda_1(N)$. This peculiar behaviour of the type II solution suggests that it cannot be identified with the Dashen and Frishman fixed point, which exists for any value of N. Even though this behaviour could be another consequence of the truncation, it is hard to justify it because for low N only operators with few spinor fields are allowed by the Pauli principle, and thus we expect our six-fermions truncation to be accurate. We have not been able to solve this puzzle.

It is easy to find many other solutions, especially for low N. For some of them there exists a minimum, either for η or for λ_1 but in other cases both curves are monotonic in w. They have also different behaviours as $N \to \infty$. We show one example in Fig. 4.

5.4. $N = 1$

Finally we will consider the special case $N = 1$. For this particular value of N not all the operators presented in Appendix A are independent. As we mentioned in Sect. 3 the effect of the Fierz transformations is to relate covariant $U(N)$ local operators (like $\overline{\psi}^a(p_1)\,\psi^b(p_2)$) to scalar ones (like $\overline{\psi}^a(p_1)\,\psi^a(p_2)$). So in the $N = 1$ case the Fierz transformations uncover relations between the S, P and the V^j operators. For example, for operators without derivatives we have the identities

$$S_{12}S_{34} = -P_{12}P_{34} = -\frac{1}{2}V^j_{12}V^j_{34}. \tag{5.11}$$

They establish relations between the coupling constants that permit us to reduce considerably the system. For the set of couplings of the four fermions operators, the independent ones are

$$\begin{aligned}
&\tilde{g} = g_1 - g_2, \qquad \tilde{u}_1 = m_1 - m_2 + m_3 - r_1 + r_2 - r_3 + (s_1 - s_2 + s_3 + s_4)/2, \\
&\tilde{u}_2 = 2m_1 + 2m_2 - 2m_3 + 2r_1 + 2r_2 - 2r_3, \qquad \tilde{u}_3 = 4m_2 + 2r_1 - 2r_2 - 2r_3 - 2s_3 - 2s_4, \\
&\tilde{u}_4 = -s_1 - s_2 + s_3 + s_4, \qquad \tilde{u}_5 = -2s_3 + 2s_4 + 2t.
\end{aligned} \tag{5.12}$$

Eq. (5.1) is now a 7-equation system that looks like

$$\begin{aligned}
&0 = 2\tilde{g}(4\eta - 3\eta^2 + 4\tilde{u}_2 w - 4\tilde{u}_3 w + 4\tilde{u}_4 w)/(4 - 3\eta) \\
&0 = 2(8\tilde{g}^2 + 4\tilde{u}_1 + 8\tilde{g}\tilde{u}_1 + 2\tilde{g}\tilde{u}_3 + 2\tilde{g}\tilde{u}_4 - 7\tilde{u}_1\eta + 3\tilde{u}_1\eta^2)/(-4 + 3\eta) \\
&0 = 2(8\tilde{g}^2 + 8\tilde{g}\tilde{u}_1 + 4\tilde{u}_2 + 8\tilde{g}\tilde{u}_4 + 4\tilde{g}\tilde{u}_5 - 7\tilde{u}_2\eta + 3\tilde{u}_2\eta^2)/(-4 + 3\eta) \\
&0 = 2(-24\tilde{g}^2 - 24\tilde{g}\tilde{u}_1 + 4\tilde{g}\tilde{u}_2 + 4\tilde{u}_3 - 12\tilde{g}\tilde{u}_3 + 4\tilde{g}\tilde{u}_5 - 7\tilde{u}_3\eta + 3\tilde{u}_3\eta^2)/(-4 + 3\eta) \\
&0 = 2(8\tilde{g}^2 + 8\tilde{g}\tilde{u}_1 + 4\tilde{g}\tilde{u}_3 + 4\tilde{u}_4 + 4\tilde{g}\tilde{u}_4 - 7\tilde{u}_4\eta + 3\tilde{s}\eta^2)/(-4 + 3\eta) \\
&0 = 2(-4\tilde{g}\tilde{u}_2 - 4\tilde{g}\tilde{u}_3 - 8\tilde{g}\tilde{u}_4 + 4\tilde{u}_5 - 7\tilde{u}_5\eta + 3\tilde{u}_5\eta^2)/(-4 + 3\eta) \\
&\eta = -2\tilde{u}_2 z + 2\tilde{u}_3 z - 4\tilde{u}_4 z + 2\tilde{u}_5 z.
\end{aligned} \tag{5.13}$$

It is linear in $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, $\tilde{u}_4$ and $\tilde{u}_5$, so we can solve it for these variables ending with a two-equation system for $\tilde{g}$ and η. After a bit of algebra and discarding the trivial solution we finally get a unique equation for η,

$$\begin{aligned}
0 = &-120w^2 z + 288wz^2 + \eta(13w^2 - 132wz + 210w^2 z + 288z^2 - 720wz^2) \\
&+ \eta^2(99wz - 90w^2 z - 432z^2 + 594wz^2) + \eta^3(162z^2 - 162wz^2).
\end{aligned} \tag{5.14}$$

As in the previous analysis we have to choose some particular scheme, i.e. fix w and z, and solve the equation numerically. Unfortunately, a simple inspection of the equation reveals bad news. The system is not undetermined and there is no room for a free $\tilde{g}$-dependence of the fixed point solution as it is true in the Thirring model. This property is satisfied in the previous order approximation (terms with less than three derivatives) where η vanishes identically. The reason this property is lost in the three derivatives approximation is unclear for us. For a more detailed analysis it is necessary to go to the next order to see whether this property is restored.

We solved Eq. (5.14) numerically for different values of w and z. As in previous examples the fixed point solutions are almost linear in the parameter z so it is more interesting to study the behaviour of the solution as a function of w. However, in the range of values studied, we did not find any non-monotonoic behaviour either in the critical couplings or in the anomalous dimension. We present some of the results in Table II.

	$w = -0.1$	$w = -0.5$	$w = -1.0$	$w = -2.0$
$z = 0.1$	1.763	3.691	6.316	11.747
$z = 0.5$	1.418	1.790	1.418	3.388
$z = 1.0$	1.376	1.559	1.811	2.349
$z = 2.0$	1.354	1.445	1.569	1.834

Table II: Values of η for $N = 1$ and different scheme parameters z and w.

6. Summary and discussions

In this article, we analyze the application of the ERG method to fermionic theories. An ERG equation for Grassmann variables is derived and the critical properties of the chiral Gross-Neveu model in two dimensions are studied with it.

To solve the ERG equation, a non-linear functional differential equation, we perform a double truncation, in the number of derivatives (derivative expansion) and in the number of fields (polynomial approximation). Unfortunately, these approximations produce similar problems that already appear in the scalar case within analogous truncations: spurious solutions and unphysical scheme dependencies. The latter, which is a common feature of almost any approximation in QFT, can be partially disentangled by invoking a minimum sensitivity criterion: for a given observable we choose the scheme that gives the most "stable" result. The emergence of spurious solutions is a more serious problem. In principle we do not have any strong argument to accept or reject a solution, except for those which lead to absurd results.

Note that for the bosonic case, within the derivative expansion, we can either expand the action as a polynomial in the fields, leading to a system of coupled non-linear equations or not make any further approximation and consider the potentials as arbitrary functions (not necessarily real analytical), that requires the study of partial differential equations. While the first approach produces lots of spurious solutions of the fixed point equations [17,18], the former has shown to produce the correct ones [15,19]. For the fermionic case, however, the situation is quite different. For finite N, within the derivative expansion, a truncation in the number of fields is not an approximation for local Lagrangians, but the definition of a function in terms of Grassmann variables. So, for fermions, the polynomial approximation should not produce fictitious solutions if we are constraining the number of derivatives. In accordance with this, for $N = 1$, which is the only case where we actually have a pure derivative expansion without any further truncation, no spurious solution appears. (There are only three solutions, besides the trivial one, and two of them have complex coupling constants, thus being rejected at once). It would be interesting to perform a pure derivative expansion for, say, $N = 2$ and check if the above pattern holds.

The first analysis we do of the fixed point structure of the chiral Gross-Neveu model is in the large N expansion of the β-functions. We find that it can be defined in two different ways, with remarkably different results. The first one leads to a continuous family of fixed-points which reminds that of Dashen and Frishman: the solution is free in the direction associated to the abelian degrees of freedom and fixed of order $1/N$ in the direction of the $SU(N)$ ones. However it presents an important difference: the anomalous dimension vanishes at leading order in contrast to the order 1 value of the Dashen and Frishman solution. We attribute this difference to the truncation. The inclusion of more terms should clarify this point.

The other type of solution is much more involved. Its anomalous dimension is non-zero but the Thirring like excitations do not appear. Moreover, unlike the preceding case its dominant eigenvalue of the linearized RG transformation depends on the scheme. Another astonishing result is the structure of the remaining eigenvalues. One would expect that the most irrelevant ones would not be too different from the canonical ones due to our truncation, which is not the case. Furthermore, the stability of the solution for finite N seems to indicate that it is not an artifact of the truncation, but a true fixed point.

To go further we proceed numerically. We can clearly identify a solution that asymptotically matches the first of the above ones and trace it to very low values of N. It presents two important drawbacks. The first one is that it is isolated, unlike the strict limit $N \to \infty$, where a one-parameter space of solutions appears. The other one is again a remaining dependence on a parameter which labels different schemes, although an accurate analysis of the most relevant critical exponent exhibits minimum sensitivity to some schemes. A search of the behaviour of the critical exponents as a function of N clearly shows that the value of λ_1 decreases with N while $N\eta$ increases.

We can also find another set of solutions, for $N > 142$ that matches the second type of the large N ones. At $N = 142.8$ it merges with another family of fixed points, with divergent β-functions when $N \to \infty$, although its critical exponents seem to be finite in that limit. This odd behaviour ruled out an identification of this solution with that of Dashen and Frishman. The lack of information in the literature (anomalous dimensions, critical exponents) about the extra fixed points of the model does not allow us to recognize our solution as any of them.

Finally, due to its peculiarities, we separately analyze the $N = 1$ case. The results are, however, discouraging. On one hand, at first order in the derivative expansion, there appears a solution with a free parameter, which labels the Thirring like excitations, but it gets spoiled when higher orders are considered. On the other hand, we find a severe two-parameter scheme dependence, which made unreliable any conclusion.

Let us remark that the insufficient non-perturbative studies (lattice computation, etc.) of the chiral Gross-Neveu model in $d = 2$ impede to discriminate definitively in favour of our results. However we are very confident of some of our findings: as we argue above, within the $1/N$ expansion the first fixed point solution is an excellent candidate for the Dashen-Frishman fixed point, whereas the other one presents evidences to be a new fixed point, with quite intricate properties, not discussed previously in the literature. Moreover, both solutions have a smooth behaviour for finite N.

We want to pause here to make a comment about redundant operators, that is, those operators that do not affect correlation functions [11]. Examples are functionals proportional to the equations of motion. They are known to be present in general and, specifically, one should expect the existence of a redundant operator that reflects the freedom of changing the normalization of our action (e.g. modifying the coefficient of the kinetic term). Its associate eigenvalue is scheme dependent and of no physical relevance. However, the ERG equation we consider presents an invariance under the normalization of the fields [14] which is not preserved by our approximation (not even in a pure derivative expansion). The signal of its restoration must be an appropriate redundant operator with vanishing eigenvalue. Its appearance should, fortunately, fix the anomalous dimension close to its correct value. It should also be mentioned that the presence of this operator has been used several times in the literature to discriminate among different schemes and find the appropriate anomalous dimension [41,16]. We expect that similar techniques should apply for fermions also, although we have not gone through the details. More work has to be done in this direction.

As a summary, we may say that, in general, our results seem somewhat discouraging, specially those for low N. Let us note, however, that the two dimensional world is rather peculiar, due to the importance of quantum effects, which generally produce large anomalous dimensions. Thus, technical simplicity turns into increasing complexity while the dimension is lowered. Nevertheless, we have gone much further than similar computations for bosonic theories, where in $d = 2$ it seems that the method completely breaks down [17]. Another interesting feature is the seemingly good results for the large N limit. At the computational level this is related to the fact that at the leading order of the $1/N$ expansion the system (5.1) of the flow equations simplifies dramatically and no room is left for spurious solutions. Actually, improvement of results in the large N limit appears to be a general feature of the ERG approach (see, for instance, Ref. [10]). Therefore, the credibility of the method can only be clearly decided after extensions to other dimensions and, possibly, the inclusion of higher order terms.

A last comment is dedicated to further work. As we have just mentioned, the formalism should be extended to higher dimensions. Equation (2.10) is prepared for that. What has to be done is to choose an appropriate action and perform a similar calculation. The number of spinor structures will be increased and, therefore, one will have to handle more terms in the action. However, we guess that, due to the greater complexity, two derivatives may be sufficient to obtain interesting results, or at least, according to the standard rule that quantum effects become less important when the dimension is increased, we hope that, within the same approximation, the results will be more transparent.

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Appendix A. Conventions and notation

In this appendix we summarize our notational conventions.

We are working always on Euclidean momentum space. A standard term in the action should be

$$\int_{p_1}\int_{p'_1}\cdots\int_{p_n}\int_{p'_n}\frac{1}{(2\pi)^{2nd}}\delta_{p_1+p'_1+\ldots+p_n+p'_n}\overline{\psi}^{a_1}_{p_1}\Gamma_1\psi^{b_1}_{p'_1}\ldots\overline{\psi}^{a_n}_{p_n}\Gamma_n\psi^{b_n}_{p'_n}\times(\text{polyn. in momenta})\times(\text{flavour matrices}), \tag{A.1}$$

where

$$\int_p \equiv \int d^dp\ , \qquad \delta_{p_1+p'_1+\ldots+p_n+p'_n} \equiv (2\pi)^d\delta^d\left(\sum_{i=1}^n (p_i+p'_i)\right)\ , \tag{A.2}$$

the fermions are denoted indicating its momentum label as a subindex, its flavour label as a superindex (and usually using Latin letters of the begining of the alphabet) and the "Clifford" label supressed; Γ is a

matrix that acts on Dirac indices (if it carries a Lorentz index we denote it with a Latin index of the middle of the alphabet). Powers of momenta (corresponding to derivatives in position space) are always indicated explicitly. For instance, the usual kinetic term would be

$$\frac{1}{(2\pi)^{2d}}\int_p\int_{p'}\overline{\psi}^a_p i\gamma^j p'^j\psi^a_{p'}\delta_{p+p'}=\frac{1}{(2\pi)^d}\int_p\overline{\psi}^a_{-p}\gamma^j p^j\psi^a_p \tag{A.3}$$

We repeatedly use Fierz transformations in order to reduce bilinears to the diagonal in flavour indices form,

$$S_{12}\equiv\overline{\psi}^a_{p_1}\psi^a_{p_2},\quad P_{12}\equiv\overline{\psi}^a_{p_1}\gamma_s\psi^a_{p_2},\quad V^j_{12}\equiv\overline{\psi}^a_{p_1}\gamma^j\psi^a_{p_2}. \tag{A.4}$$

When we do so all flavour matrices disappear.

Even more, for the sake of simplicity, we often drop the integral signs, the delta functions and the powers of $(2\pi)^d$. However, when a term of the action is written they should always be understood to be present. Also, for the action in Appendix B it is convenient to define the sum and difference of the momenta of every bilinear. For instance, the above kinetic term will be written as

$$\frac{1}{2}p^{-\,j}_{12}iV^j_{12} \tag{A.5}$$

with $p^{\pm\,j}_{mn}\equiv(p_m\pm p_n)^j$ and with the integrals and momentum conservation delta function being assumed.

The conventions for the Clifford algebra are the usual ones on the two-dimensional Euclidean space [39],

$$\left\{\gamma^i,\gamma^j\right\}=2\delta^{ij},\quad \gamma_s=-i\gamma^1\gamma^2. \tag{A.6}$$

The completely antisymmetric tensor ϵ^{ij} is $\epsilon^{12}=-\epsilon^{21}=1$.

We always use N to indicate the number of flavours and η to denote the anomalous dimension of the fields. The conventions for the coupling constants are a Latin or Greek letter with a subindex. The letters are: 1) g for the four-fermions, non-derivative operators,

$$g_1(S_{12}S_{34}-P_{12}P_{34})+g_2V^j_{12}V^j_{34}\,; \tag{A.7}$$

2) m, r, s and t for the various types of four-fermions, two-derivatives operator couplings; 3) a, c and e for the three types of six-fermions, one-derivative operator couplings; 4) Greek indices κ, ι, υ, ε and ς for the six-fermions and three-derivatives operator couplings.

When we expand the full action we introduce the scheme-dependent parameters

$$\alpha=\frac{1}{(2\pi)^d}\int_p K'\left(p^2\right),\quad \beta=\frac{1}{(2\pi)^d}\int_p p^2K'\left(p^2\right),\quad \gamma=K'\left(0\right),\quad \delta=K''\left(0\right). \tag{A.8}$$

Sometimes they enter solely in the combinations $\omega=\frac{\beta\delta}{\alpha\gamma}$, $z=\frac{\delta}{\gamma^2}$.

A dot in any quantity means a derivative with respect to the RG flow parameter t.

Appendix B. The action

In this appendix we present the complete action we use for the computation of the β-functions. For the sake of clarity the action is divided into subactions according to the number of fermions and the number of derivatives, and in the case of six fermions and three derivatives also according to the fermionic structure. The integrals over the momenta and the δ-functions of global momentum conservation (with their respective powers of $(2\pi)^d$) are always omitted. The notation is presented in the Appendix A.

$$
\begin{aligned}
\mathcal{S}^{(2,1)} &= \frac{1}{2} p_{12}^{-\,j} i V_{12}^{j}. \\
\mathcal{S}^{(4,0)} &= g_1(S_{12}S_{34} - P_{12}P_{34}) + g_2 V_{12}^{j} V_{34}^{j}. \\
\mathcal{S}^{(4,2)} &= \{m_1 p_{12}^{+\,2} + m_2 p_{12}^{-} \cdot p_{34}^{-} + m_3 p_{12}^{-\,2}\} \times (S_{12}S_{34} - P_{12}P_{34}) + \{r_1 p_{12}^{+\,2} + r_2 p_{12}^{-} \cdot p_{34}^{-} + r_3 p_{12}^{-\,2}\} \times V_{12}^{j} V_{34}^{j} \\
&\quad + \{s_1 p_{34}^{+\,j} p_{12}^{+\,k} + s_2 p_{12}^{-\,j} p_{12}^{-\,k} + s_3 p_{12}^{-\,j} p_{34}^{-\,k} + s_4 p_{34}^{-\,j} p_{12}^{-\,k}\} \times V_{12}^{j} V_{34}^{k} + t p_{34}^{+\,j} p_{34}^{-\,k} \epsilon^{jk} (S_{12}P_{34} - P_{12}S_{34}). \\
\mathcal{S}^{(6,1)} &= \{a_1 p_{12}^{-\,j} + a_2 p_{56}^{-\,j}\} \times i(S_{12}S_{34} - P_{12}P_{34}) V_{56}^{j} + \{c_1 p_{12}^{-\,j} + c_2 p_{56}^{-\,j}\} \times i V_{12}^{k} V_{34}^{k} V_{56}^{j} \\
&\quad + e_1 p_{12}^{+\,k} \epsilon^{jk} i (P_{12}S_{34} - S_{12}P_{34}) V_{56}^{j}. \\
\mathcal{S}^{(6,3)a} &= \{\kappa_1 p_{12}^{-\,j} p_{12}^{-} \cdot p_{12}^{-} + \kappa_2 p_{12}^{-\,j} p_{12}^{-} \cdot p_{34}^{-} + \kappa_3 p_{12}^{-\,j} p_{34}^{+} \cdot p_{34}^{+} + \kappa_4 p_{12}^{-\,j} p_{34}^{+} \cdot p_{56}^{+} + \kappa_5 p_{12}^{-\,j} p_{34}^{-} \cdot p_{34}^{-} \\
&\quad + \kappa_6 p_{12}^{-\,j} p_{34}^{-} \cdot p_{56}^{-} + \kappa_7 p_{34}^{+\,j} p_{12}^{-} \cdot p_{34}^{+} + \kappa_8 p_{34}^{+\,j} p_{12}^{-} \cdot p_{56}^{+} + \kappa_9 p_{34}^{+\,j} p_{34}^{+} \cdot p_{34}^{-} + \kappa_{10} p_{34}^{+\,j} p_{34}^{+} \cdot p_{56}^{-} \\
&\quad + \kappa_{11} p_{34}^{+\,j} p_{34}^{-} \cdot p_{56}^{+} + \kappa_{12} p_{34}^{+\,j} p_{56}^{+} \cdot p_{56}^{-} + \kappa_{13} p_{34}^{-\,j} p_{12}^{-} \cdot p_{12}^{-} + \kappa_{14} p_{34}^{-\,j} p_{12}^{-} \cdot p_{34}^{-} + \kappa_{15} p_{34}^{-\,j} p_{12}^{-} \cdot p_{56}^{-} \\
&\quad + \kappa_{16} p_{34}^{-\,j} p_{34}^{+} \cdot p_{34}^{+} + \kappa_{17} p_{34}^{-\,j} p_{34}^{+} \cdot p_{56}^{+} + \kappa_{18} p_{34}^{-\,j} p_{34}^{-} \cdot p_{34}^{-} + \kappa_{19} p_{34}^{-\,j} p_{34}^{-} \cdot p_{56}^{-} + \kappa_{20} p_{34}^{-\,j} p_{56}^{+} \cdot p_{56}^{+} \\
&\quad + \kappa_{21} p_{34}^{-\,j} p_{56}^{-} \cdot p_{56}^{-}\} \times i V_{12}^{j} (S_{34}S_{56} - P_{34}P_{56}). \\
\mathcal{S}^{(6,3)b} &= \{\iota_1 p_{12}^{-\,j} p_{12}^{-} \cdot p_{12}^{-} + \iota_2 p_{12}^{-\,j} p_{12}^{-} \cdot p_{34}^{-} + \iota_3 p_{12}^{-\,j} p_{34}^{+} \cdot p_{34}^{+} + \iota_4 p_{12}^{-\,j} p_{34}^{+} \cdot p_{56}^{+} + \iota_5 p_{12}^{-\,j} p_{34}^{-} \cdot p_{34}^{-} \\
&\quad + \iota_6 p_{12}^{-\,j} p_{34}^{-} \cdot p_{56}^{-} + \iota_7 p_{34}^{+\,j} p_{12}^{-} \cdot p_{34}^{+} + \iota_8 p_{34}^{+\,j} p_{12}^{-} \cdot p_{56}^{+} + \iota_9 p_{34}^{+\,j} p_{34}^{+} \cdot p_{34}^{-} + \iota_{10} p_{34}^{+\,j} p_{34}^{+} \cdot p_{56}^{-} \\
&\quad + \iota_{11} p_{34}^{+\,j} p_{34}^{-} \cdot p_{56}^{+} + \iota_{12} p_{34}^{+\,j} p_{56}^{+} \cdot p_{56}^{-} + \iota_{13} p_{34}^{-\,j} p_{12}^{-} \cdot p_{12}^{-} + \iota_{14} p_{34}^{-\,j} p_{12}^{-} \cdot p_{34}^{-} + \iota_{15} p_{34}^{-\,j} p_{12}^{-} \cdot p_{56}^{-} \\
&\quad + \iota_{16} p_{34}^{-\,j} p_{34}^{+} \cdot p_{34}^{+} + \iota_{17} p_{34}^{-\,j} p_{34}^{+} \cdot p_{56}^{+} + \iota_{18} p_{34}^{-\,j} p_{34}^{-} \cdot p_{34}^{-} + \iota_{19} p_{34}^{-\,j} p_{34}^{-} \cdot p_{56}^{-} + \iota_{20} p_{34}^{-\,j} p_{56}^{+} \cdot p_{56}^{+} \\
&\quad + \iota_{21} p_{34}^{-\,j} p_{56}^{-} \cdot p_{56}^{-}\} \times i V_{12}^{j} (S_{34}S_{56} - P_{34}P_{56}). \\
\mathcal{S}^{(6,3)c} &= \{\upsilon_1 p_{12}^{-\,j} p_{12}^{+\,k} p_{12}^{+\,l} + \upsilon_2 p_{12}^{-\,j} p_{12}^{-\,k} p_{12}^{-\,l} + \upsilon_3 p_{12}^{-\,j} p_{12}^{+\,k} p_{34}^{+\,l} + \upsilon_4 p_{12}^{-\,j} p_{12}^{-\,k} p_{34}^{-\,l} + \upsilon_5 p_{12}^{-\,j} p_{12}^{-\,k} p_{56}^{-\,l} \\
&\quad + \upsilon_6 p_{34}^{-\,j} p_{12}^{+\,k} p_{12}^{+\,l} + \upsilon_7 p_{34}^{+\,j} p_{12}^{+\,k} p_{12}^{-\,l} + \upsilon_8 p_{56}^{+\,j} p_{12}^{+\,k} p_{12}^{-\,l} + \upsilon_9 p_{34}^{-\,j} p_{12}^{-\,k} p_{12}^{-\,l} + \upsilon_{10} p_{12}^{-\,j} p_{34}^{-\,k} p_{56}^{-\,l} \\
&\quad + \upsilon_{11} p_{12}^{-\,j} p_{56}^{+\,k} p_{34}^{+\,l} + \upsilon_{12} p_{12}^{-\,j} p_{56}^{-\,k} p_{34}^{-\,l} + \upsilon_{13} p_{56}^{+\,j} p_{12}^{+\,k} p_{34}^{-\,l} + \upsilon_{14} p_{56}^{-\,j} p_{12}^{-\,k} p_{34}^{-\,l}\} \times i V_{12}^{j} V_{34}^{k} V_{56}^{l}. \\
\mathcal{S}^{(6,3)d} &= \{\varepsilon_1 p_{12}^{+\,j} p_{12}^{+} \cdot p_{12}^{+} + \varepsilon_2 p_{12}^{+\,j} p_{12}^{+} \cdot p_{34}^{+} + \varepsilon_3 p_{12}^{+\,j} p_{34}^{+} \cdot p_{34}^{+} + \varepsilon_4 p_{12}^{+\,j} p_{12}^{-} \cdot p_{12}^{-} + \varepsilon_5 p_{12}^{+\,j} p_{12}^{-} \cdot p_{34}^{-} \\
&\quad + \varepsilon_6 p_{12}^{+\,j} p_{12}^{-} \cdot p_{56}^{-} + \varepsilon_7 p_{12}^{+\,j} p_{34}^{-} \cdot p_{34}^{-} + \varepsilon_8 p_{12}^{+\,j} p_{34}^{-} \cdot p_{56}^{-} + \varepsilon_9 p_{12}^{+\,j} p_{56}^{-} \cdot p_{56}^{-} + \varepsilon_{10} p_{12}^{-\,j} p_{12}^{+} \cdot p_{12}^{-} \\
&\quad + \varepsilon_{11} p_{12}^{-\,j} p_{12}^{+} \cdot p_{34}^{-} + \varepsilon_{12} p_{12}^{-\,j} p_{12}^{+} \cdot p_{56}^{-} + \varepsilon_{13} p_{12}^{-\,j} p_{34}^{+} \cdot p_{12}^{-} + \varepsilon_{14} p_{12}^{-\,j} p_{34}^{+} \cdot p_{34}^{-} + \varepsilon_{15} p_{12}^{-\,j} p_{34}^{+} \cdot p_{56}^{-} \\
&\quad + \varepsilon_{16} p_{56}^{-\,j} p_{12}^{+} \cdot p_{12}^{-} + \varepsilon_{17} p_{56}^{-\,j} p_{12}^{+} \cdot p_{34}^{-} + \varepsilon_{18} p_{56}^{-\,j} p_{12}^{+} \cdot p_{56}^{-}\} \times i \epsilon^{kj} (S_{12}P_{34} - P_{12}S_{34}) V_{56}^{k}. \\
\mathcal{S}^{(6,3)e} &= \{\varsigma_1 p_{12}^{+\,j} p_{12}^{+\,k} p_{34}^{+\,l} + \varsigma_2 p_{12}^{+\,j} p_{12}^{-\,k} p_{34}^{-\,l} + \varsigma_3 p_{12}^{+\,j} p_{12}^{-\,k} p_{56}^{-\,l} + \varsigma_4 p_{12}^{+\,j} p_{34}^{-\,k} p_{56}^{-\,l} + \varsigma_5 p_{12}^{-\,j} p_{12}^{+\,k} p_{12}^{-\,l} \\
&\quad + \varsigma_6 p_{12}^{-\,j} p_{12}^{+\,k} p_{34}^{-\,l} + \varsigma_7 p_{12}^{-\,j} p_{12}^{+\,k} p_{56}^{-\,l} + \varsigma_8 p_{12}^{-\,j} p_{34}^{+\,k} p_{12}^{-\,l} + \varsigma_9 p_{12}^{-\,j} p_{34}^{+\,k} p_{34}^{-\,l} + \varsigma_{10} p_{12}^{-\,j} p_{34}^{+\,k} p_{56}^{-\,l} \\
&\quad + \varsigma_{11} p_{56}^{-\,j} p_{12}^{+\,k} p_{12}^{-\,l} + \varsigma_{12} p_{56}^{-\,j} p_{12}^{+\,k} p_{34}^{-\,l} + \varsigma_{13} p_{56}^{-\,j} p_{12}^{+\,k} p_{56}^{-\,l}\} \times i \epsilon^{kl} (S_{12}P_{34} - P_{12}S_{34}) V_{56}^{j}.
\end{aligned} \tag{B.1}
$$

The conventions to get rid of non-independent operators are as follows. In $\mathcal{S}^{(4,2)}$ we consider terms containing $p_{34}^{+\,k}$ multiplied by neither $(S_{12}S_{34} - P_{12}P_{34})$ nor $V_{12}^{j}V_{34}^{j}$, because, for instance, $p_{12}^{+} \cdot p_{34}^{+} V_{12}^{j} V_{34}^{j} =$

$-p_{12}^{+\,2} V_{12}^j V_{34}^j$. We do not take into account operators containing $(S_{12}S_{34} - P_{12}P_{34})V_{56}^j p_{56}^{+\,k}$, $V_{12}^k V_{34}^k V_{56}^j p_{56}^{+\,l}$ or $(S_{12}P_{34} - P_{12}S_{34})V_{56}^j p_{56}^{+\,k}$ either. Finally, for terms like $V_{12}^j V_{34}^k$ we integrate by parts if they are multiplied by $p_{12}^{+\,j}$ or $p_{34}^{+\,k}$, e.g., $V_{12}^j V_{34}^k p_{12}^{+\,j} p_{34}^{+\,k} = -V_{12}^j V_{34}^k p_{34}^{+\,j} p_{34}^{+\,k} = V_{12}^j V_{34}^k p_{34}^j p_{12}^{+\,k}$, and similarly, for $V_{12}^j V_{34}^k V_{56}^l$ multiplied by $p_{12}^{+\,j}$, $p_{34}^{+\,k}$ or $p_{56}^{+\,l}$.

We do not include $\mathcal{S}^{(2,3)} = iV_{12}^j (p_1 - p_2)^j (p_1 - p_2)^2$ as we have discussed in Sect. 3. Without it, we have a basis of 106 independent operators.

Appendix C. β functions

In this appendix we present the complete set of β-functions. The scheme dependent parameters α, β, γ and δ have been defined in Eq. (5.3).

$$\eta = 4\alpha[-m_1 + m_2 + m_3 + s_1 + s_2 + s_3 + s_4 + t - 2N(r_2 + 2s_3 + s_4)],$$

$$\begin{aligned}
\dot{g}_1 &= 2g_1\eta + 2\alpha(a_1 + a_2 + e_1 - a_2 N) + 2\beta(-\varepsilon_1 - \varepsilon_4 - \varepsilon_6 - \varepsilon_9 - \varepsilon_{10} - \varepsilon_{12} - \varepsilon_{16} - \varepsilon_{18} \\
&\quad + 2\kappa_1 + \kappa_2 + \kappa_3 + \kappa_5 + \kappa_7 + \kappa_9 + \kappa_{13} + \kappa_{14} + \kappa_{16} + \kappa_{18} - 8\kappa_1 N), \\
\dot{g}_2 &= 2g_2\eta + 2\alpha(a_1 + c_1 + c_2 + e_1 - c_1 N - c_2 N) + \beta[2(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_7 - \varepsilon_{10} - \varepsilon_{11} + \varepsilon_{13} + \\
&\quad \varepsilon_{14}) + 2\upsilon_1 + 2\upsilon_2 - \upsilon_3 + \upsilon_4 + \upsilon_5 + \upsilon_6 - \upsilon_7 - \upsilon_8 + \upsilon_9 + 2(\kappa_9 + \kappa_{10} - \kappa_{11} - \kappa_{12} + \kappa_{16} - \kappa_{17} + \kappa_{18} \\
&\quad + \kappa_{19} + \kappa_{20} + \kappa_{21}) + 2(2\iota_1 + \iota_{13} + \iota_{14} + \iota_{16} + \iota_{18} + \iota_2 + \iota_3 + \iota_5 + \iota_7 + \iota_9) - 8N(\upsilon_2 + 2\iota_1 + \iota_{18})], \\
\dot{m}_1 &= (-2 + 2\eta)m_1 + \frac{\alpha}{2}[-2\varepsilon_1 + 5\varepsilon_2 - 6\varepsilon_3 + \varepsilon_6 + 2\varepsilon_7 - 4\varepsilon_9 + \varepsilon_{10} - 3\varepsilon_{13} + \varepsilon_{15} + \varepsilon_{16} + \varepsilon_{17} - 5\varepsilon_{18} \\
&\quad + \varsigma_1 + \varsigma_3 + 2\varsigma_4 + \varsigma_5 - \varsigma_8 - \varsigma_{10} - \varsigma_{11} - \varsigma_{12} - \varsigma_{13} + 4\kappa_1 - \kappa_2 + 6\kappa_3 - 6\kappa_4 + 4\kappa_7 - 5\kappa_8 - \kappa_9 \\
&\quad + 2\kappa_{10} + \kappa_{11} - \kappa_{14} - \kappa_{17} + 2\kappa_{18} + 4\kappa_{20} + 4N(-2\kappa_3 + 2\kappa_4 - \kappa_7 + \kappa_8)], \\
\dot{m}_2 &= (-2 + 2\eta)m_2 + \frac{\alpha}{2}[-\varepsilon_5 - \varepsilon_8 + \varepsilon_{11} + \varepsilon_{17} - \varsigma_2 + \varsigma_4 + \varsigma_6 - 2\varsigma_9 - 2\varsigma_{10} + \varsigma_{12} \\
&\quad + 3\kappa_2 + 6\kappa_6 - 3\kappa_{10} + 2\kappa_{13} + 5\kappa_{15} + 3\kappa_{19} - 2\kappa_{20} + 2\kappa_{21} - 4N(2\kappa_6 + \kappa_{15})], \\
\dot{m}_3 &= (-2 + 2\eta)m_3 + \frac{\alpha}{2}[-2\varepsilon_1 - 4\varepsilon_7 + \varepsilon_{10} + \varepsilon_{12} + 2\varepsilon_{13} + \varepsilon_{16} + \varepsilon_{18} + \varsigma_5 + \varsigma_7 - 2\varsigma_8 + \varsigma_{11} + \varsigma_{13} \\
&\quad + 4\kappa_1 + 2\kappa_2 + 6\kappa_5 - \kappa_7 - \kappa_9 + 2\kappa_{13} + 4\kappa_{14} + 2\kappa_{18} + 2\kappa_{19} + 4\kappa_{21} - 4N(2\kappa_5 + \kappa_{14})], \\
\dot{r}_1 &= (-2 + 2\eta)r_1 + \frac{\alpha}{4}[2(-3\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - 3\varepsilon_7 + \varepsilon_{10} + \varepsilon_{11} + 3\varepsilon_{13} - \varepsilon_{14}) \\
&\quad + 2\upsilon_1 - 2\upsilon_2 - 3\upsilon_3 - \upsilon_4 + \upsilon_5 - 5\upsilon_6 - 3\upsilon_7 + 3\upsilon_8 + \upsilon_9 + 4\upsilon_{11} + 4\upsilon_{13} \\
&\quad + 2(-\kappa_9 + \kappa_{10} - \kappa_{11} + \kappa_{12} + \kappa_{16} + \kappa_{17} + \kappa_{18} - \kappa_{19} + \kappa_{20} + \kappa_{21}) + 2(4\iota_1 - \iota_2 + 6\iota_3 - 6\iota_4 \\
&\quad + 4\iota_7 - 5\iota_8 - \iota_9 + 2\iota_{10} + \iota_{11} - \iota_{14} - \iota_{17} + 2\iota_{18} + 4\iota_{20}) + 8N(-2\iota_3 + 2\iota_4 - \iota_7 + \iota_8 - \iota_{20})], \\
\dot{r}_2 &= (-2 + 2\eta)r_2 + \frac{\alpha}{2}[-\upsilon_4 + \upsilon_6 - \upsilon_9 - 2\upsilon_{12} - \upsilon_{13} - 3\upsilon_{14} + 2(-\kappa_7 + \kappa_8 + \kappa_{14} + \kappa_{15}) \\
&\quad + 3\iota_2 + 6\iota_6 - 3\iota_{10} + 2\iota_{13} + 5\iota_{15} + 3\iota_{19} - 2\iota_{20} + 2\iota_{21} - 8N(\iota_6 + \iota_{15})], \\
\dot{r}_3 &= (-2 + 2\eta)r_3 + \frac{\alpha}{4}[2(-3\varepsilon_1 + 3\varepsilon_2 - 3\varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_7 - 4\varepsilon_9 + \varepsilon_{10} + \varepsilon_{11} - \varepsilon_{13} - \varepsilon_{14} - 4\varepsilon_{18}) + 2\upsilon_1 - 2\upsilon_2 \\
&\quad - \upsilon_3 - \upsilon_4 - \upsilon_5 + \upsilon_6 + \upsilon_7 - \upsilon_8 - 5\upsilon_9 + 2(-\kappa_9 - \kappa_{10} + \kappa_{11} + \kappa_{12} + 4\kappa_{13} + \kappa_{16} - \kappa_{17} + \kappa_{18} + \kappa_{19} + \kappa_{20} \\
&\quad + \kappa_{21}) + 2(4\iota_1 + 2\iota_2 + 6\iota_5 - \iota_7 - \iota_9 + 2\iota_{13} + 4\iota_{14} + 2\iota_{18} + 2\iota_{19} + 4\iota_{21}) - 8N(2\iota_5 + \iota_{13} + \iota_{14} + \iota_{21})], \\
\dot{s}_1 &= (-2 + 2\eta)s_1 + \frac{\alpha}{2}[2(-\varepsilon_1 + \varepsilon_3 + \varepsilon_4 - \varepsilon_7 + \varepsilon_{11} + \varepsilon_{13}) - 2(\varsigma_1 + \varsigma_2 + \varsigma_5 + \varsigma_9)
\end{aligned}$$

$$
\begin{aligned}
&+2\upsilon_1-6\upsilon_2-\upsilon_3+\upsilon_4+\upsilon_5-7\upsilon_6-\upsilon_7-\upsilon_8+\upsilon_9+8\upsilon_{11}+8\upsilon_{13}\\
&+2(-\kappa_{10}-\kappa_{11}+\kappa_{16}-\kappa_{18}-\kappa_{20}+\kappa_{21})+4N(\upsilon_6-2\upsilon_{11}-\upsilon_{13}+\iota_{10})],\\
\dot{s}_2&=(-2+2\eta)s_2+\frac{\alpha}{2}[2(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_7+2\varepsilon_{18})+2(\varsigma_5+\varsigma_6-\varsigma_8-\varsigma_9+2\varsigma_{13})\\
&-2\upsilon_1+6\upsilon_2+\upsilon_3+7\upsilon_4+7\upsilon_5-\upsilon_6+\upsilon_7+\upsilon_8+7\upsilon_9\\
&+2(2\kappa_2-\kappa_{16}+\kappa_{17}+\kappa_{18}+\kappa_{19}-\kappa_{20}+\kappa_{21})-4N(\upsilon_4+2\upsilon_5+\upsilon_9+\iota_2+\iota_{19})],\\
\dot{s}_3&=(-2+2\eta)s_3+\frac{\alpha}{2}[2(-\varepsilon_6-\varepsilon_8+\varepsilon_{12}-\varepsilon_{15})+2(-\varsigma_3-\varsigma_4+\varsigma_7-\varsigma_{10}+2\varsigma_{11}+2\varsigma_{12})\\
&+\upsilon_4+4\upsilon_5-\upsilon_6+\upsilon_9+12\upsilon_{10}+4\upsilon_{11}+6\upsilon_{12}+\upsilon_{13}+3\upsilon_{14}\\
&+4(-\kappa_3+\kappa_4+\kappa_5+\kappa_6)-\iota_{10}-2\iota_{13}-\iota_{15}+\iota_{19}+\iota_2+2\iota_6+2\iota_{20}-2\iota_{21}-8N(3\upsilon_{10}+\upsilon_{12}+\iota_6)],\\
\dot{s}_4&=(-2+2\eta)s_4+\frac{\alpha}{2}[3(\upsilon_4-\upsilon_6+\upsilon_9+2\upsilon_{12}+\upsilon_{13}+3\upsilon_{14})+2(-\kappa_7+\kappa_8+\kappa_{14}+\kappa_{15})\\
&-\iota_2-2\iota_6+\iota_{10}+2\iota_{13}+\iota_{15}-\iota_{19}-2\iota_{20}+2\iota_{21}-4N(2\upsilon_{12}+3\upsilon_{14}+\iota_{15})],\\
\dot{t}&=(-2+2\eta)t+\frac{\alpha}{2}(\varepsilon_2+2\varepsilon_3-3\varepsilon_5-3\varepsilon_6-2\varepsilon_7-3\varepsilon_8-4\varepsilon_9+3\varepsilon_{11}+3\varepsilon_{12}-\varepsilon_{13}-3\varepsilon_{15}+2\varepsilon_{18}\\
&+3(-\varsigma_1-\varsigma_2-\varsigma_3-\varsigma_4+\varsigma_6+\varsigma_7-\varsigma_8-2\varsigma_9-\varsigma_{10}+2\varsigma_{11}+2\varsigma_{12}+2\varsigma_{13})\\
&+2\kappa_2-2\kappa_3+2\kappa_4+2\kappa_5+2\kappa_6+\kappa_7-\kappa_8+\kappa_{10}-3\kappa_{11}-4\kappa_{13}-\kappa_{14}-\kappa_{15}+3\kappa_{17}-\kappa_{19}-2\kappa_{20}+2\kappa_{21}\\
&+4N(\varepsilon_6+\varepsilon_8-\varepsilon_{12}+\varepsilon_{15}+\varsigma_3+\varsigma_4-\varsigma_7+\varsigma_{10}-2\varsigma_{11}-2\varsigma_{12})),
\end{aligned}
$$

$$
\begin{aligned}
&\dot{a}_1=(-2+3\eta)a_1+4g_1g_2\gamma,\quad \dot{a}_2=(-2+3\eta)a_2+2g_1^2\gamma,\quad \dot{c}_1=(-2+3\eta)c_1+4g_2^2\gamma,\\
&\dot{c}_2=(-2+3\eta)c_2-2g_2^2\gamma,\quad \dot{e}=(-2+3\eta)e-4g_1^2\gamma-4g_1g_2\gamma,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\kappa}_1&=(-4+3\eta)\kappa_1+\delta g_1^2/2+2g_1\gamma m_3,\quad \dot{\kappa}_2=(-4+3\eta)\kappa_2+2\gamma g_1(2m_2+s_2),\\
\dot{\kappa}_3&=(-4+3\eta)\kappa_3+\delta g_1^2+2\gamma g_1(2m_1+m_3+4s_3+t),\quad \dot{\kappa}_4=(-4+3\eta)\kappa_4-\delta g_1^2+2\gamma g_1(2s_3-t),\\
\dot{\kappa}_5&=(-4+3\eta)\kappa_5+2\gamma g_1(m_3+2s_3),\quad \dot{\kappa}_6=(-4+3\eta)\kappa_6\\
\dot{\kappa}_7&=(-4+3\eta)\kappa_7+2\delta g_1^2+2\gamma g_1(2m_3+4r_2+4s_4-t),\\
\dot{\kappa}_8&=(-4+3\eta)\kappa_8-2\delta g_1^2+2\gamma g_1(-2m_3+2r_2+2s_4+t),\\
\dot{\kappa}_9&=(-4+3\eta)\kappa_9+2\delta g_1g_2+2\gamma(-g_1(2s_1+t)+2g_2(m_2+m_3-t)),\\
\dot{\kappa}_{10}&=(-4+3\eta)\kappa_{10}+8\delta g_1g_2+2\gamma(g_1(2m_2+2r_3-2s_1+3s_2)+2g_2(m_2+2m_3-t)),\\
\dot{\kappa}_{11}&=(-4+3\eta)\kappa_{11}+4\delta g_1g_2+2\gamma(g_1(-2m_2+2r_3-2s_1+t)+2g_2(2m_2+m_3)),\\
\dot{\kappa}_{12}&=(-4+3\eta)\kappa_{12}+4\delta g_1g_2+2\gamma(g_1(-2s_1+s_2)+2g_2(m_2+2m_3-t)),\quad \dot{\kappa}_{13}=(-4+3\eta)\kappa_{13}+2\gamma g_1r_3,\\
\dot{\kappa}_{14}&=(-4+3\eta)\kappa_{14}+4\gamma g_1(r_2+s_4),\quad \dot{\kappa}_{15}=(-4+3\eta)\kappa_{15},\\
\dot{\kappa}_{16}&=(-4+3\eta)\kappa_{16}+\delta g_1g_2+2\gamma(g_1(2r_1+t)+g_2(m_3+2t)),\\
\dot{\kappa}_{17}&=(-4+3\eta)\kappa_{17}+4\delta g_1g_2+2\gamma(g_1(4r_1+s_2-t)+4g_2(m_3+t)),\\
\dot{\kappa}_{18}&=(-4+3\eta)\kappa_{18}+\delta g_1g_2+2\gamma(g_1(r_3+s_2)+g_2m_3),\\
\dot{\kappa}_{19}&=(-4+3\eta)\kappa_{19}+4\gamma g_2m_2,\quad \dot{\kappa}_{20}=(-4+3\eta)\kappa_{20}+4\delta g_1g_2+\gamma(g_1(2r_1+r_3+2s_2)+g_2(2m_1+m_3+2t)),\\
\dot{\kappa}_{21}&=(-4+3\eta)\kappa_{21}+2\gamma g_2m_3,\quad \dot{\iota}_1=(-4+3\eta)\iota_1-\delta g_2^2/2-2\gamma g_2r_3,\quad \dot{\iota}_2=(-4+3\eta)\iota_2+2\gamma g_2(-2r_2+s_2),\\
\dot{\iota}_3&=(-4+3\eta)\iota_3-\delta g_2^2+2\gamma g_2(-2r_1-r_3+4s_3),\quad \dot{\iota}_4=(-4+3\eta)\iota_4+\delta g_2^2+4\gamma g_2s_3,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\iota}_5 = (-4+3\eta)\iota_5 + 2\gamma g_2(-r_3+2s_3), \quad \dot{\iota}_6 = (-4+3\eta)\iota_6, \quad \dot{\iota}_7 = (-4+3\eta)\iota_7 - 2\delta g_2^2 + 4\gamma g_2(2r_2 - r_3 + 2s_4),\\
&\dot{\iota}_8 = (-4+3\eta)\iota_8 + 2\delta g_2^2 + 4\gamma g_2(r_2+r_3+s_4), \quad \dot{\iota}_9 = (-4+3\eta)\iota_9 + 2\delta g_2^2 + 4\gamma g_2(2r_2+r_3-s_1),\\
&\dot{\iota}_{10} = (-4+3\eta)\iota_{10} + 8\delta g_2^2 + 2\gamma g_2(8r_3 - 2s_1 + 3s_2), \quad \dot{\iota}_{11} = (-4+3\eta)\iota_{11} + 4\delta g_2^2 + 4\gamma g_2(3r_2 + 2r_3 - s_1),\\
&\dot{\iota}_{12} = (-4+3\eta)\iota_{12} + 4\delta g_2^2 + 2\gamma g_2(2r_2 + 4r_3 - 2s_1 + s_2), \quad \dot{\iota}_{13} = (-4+3\eta)\iota_{13} + 2\gamma g_2 r_3,\\
&\dot{\iota}_{14} = (-4+3\eta)\iota_{14} + 4\gamma g_2(r_2+s_4), \quad \dot{\iota}_{15} = (-4+3\eta)\iota_{15}, \quad \dot{\iota}_{16} = (-4+3\eta)\iota_{16} + \delta g_2^2 + 2\gamma g_2(2r_1 + r_3),\\
&\dot{\iota}_{17} = (-4+3\eta)\iota_{17} + \delta 4 g_2^2 + 2\gamma g_2(4r_1 + 2r_3 + s_2), \quad \dot{\iota}_{18} = (-4+3\eta)\iota_{18} + \delta g_2^2 + 2\gamma g_2(2r_3 + s_2),\\
&\dot{\iota}_{19} = (-4+3\eta)\iota_{19} + 4\gamma g_2 r_2, \quad \dot{\iota}_{20} = (-4+3\eta)\iota_{20} + 4g_2^2\delta + 4\gamma g_2(2r_1 + r_3 + s_2),\\
&\dot{\iota}_{21} = (-4+3\eta)\iota_{21} + 2\gamma g_2 r_3, \quad \dot{\upsilon}_1 = (-4+3\eta)\upsilon_1 - 4\gamma g_2 s_3, \quad \dot{\upsilon}_2 = (-4+3\eta)\upsilon_2,\\
&\dot{\upsilon}_3 = (-4+3\eta)\upsilon_3 - 4\gamma g_2(s_1 + 2s_3), \quad \dot{\upsilon}_4 = (-4+3\eta)\upsilon_4 + 2\gamma g_2(s_2 - 2s_4), \quad \dot{\upsilon}_5 = (-4+3\eta)\upsilon_5 - 2\gamma g_2 s_2,\\
&\dot{\upsilon}_6 = (-4+3\eta)\upsilon_6 - 2\gamma g_2(s_2 + 2s_4), \quad \dot{\upsilon}_7 = (-4+3\eta)\upsilon_7 + 4\gamma g_2(s_1 - 2s_4),\\
&\dot{\upsilon}_8 = (-4+3\eta)\upsilon_8 + 2\gamma g_2(-s_2 + 2s_4), \quad \dot{\upsilon}_9 = (-4+3\eta)\upsilon_9 + 4\gamma g_2 s_4, \quad \dot{\upsilon}_{10} = (-4+3\eta)\upsilon_{10},\\
&\dot{\upsilon}_{11} = (-4+3\eta)\upsilon_{11} - 2\gamma g_2(2s_1 + s_2), \quad \dot{\upsilon}_{12} = (-4+3\eta)\upsilon_{12}, \quad \dot{\upsilon}_{13} = (-4+3\eta)\upsilon_{13} + 4\gamma g_2(s_1 - s_4),\\
&\dot{\upsilon}_{14} = (-4+3\eta)\upsilon_{14}, \quad \dot{\varepsilon}_1 = (-4+3\eta)\varepsilon_1 + \delta g_1(g_1 + 7g_2) + 2\gamma(g_1(2m_1 + m_3 + 2r_1 + 2r_3) + g_2(4m_1 + m_3)),\\
&\dot{\varepsilon}_2 = (-4+3\eta)\varepsilon_2 + 2\delta g_1(-g_1 + 2g_2) + 4\gamma(2g_1 r_1 + g_2 m_3),\\
&\dot{\varepsilon}_3 = (-4+3\eta)\varepsilon_3 + \delta g_1(g_1 - 2g_2) + 2\gamma(g_1(2m_1 + m_3 + 2r_1 - r_3) + g_2(-2m_1 + m_3)),\\
&\dot{\varepsilon}_4 = (-4+3\eta)\varepsilon_4 - \delta g_1 g_2 + 2\gamma(g_1(m_3 - r_3) + g_2 m_3), \quad \dot{\varepsilon}_5 = (-4+3\eta)\varepsilon_5 + 4\gamma g_2 m_2,\\
&\dot{\varepsilon}_6 = (-4+3\eta)\varepsilon_6 + 4\gamma g_1(m_2 - r_2), \quad \dot{\varepsilon}_7 = (-4+3\eta)\varepsilon_7 + 2\delta g_1 g_2 + 2\gamma(g_1(m_3 + 2r_3) + g_2 m_3),\\
&\dot{\varepsilon}_8 = (-4+3\eta)\varepsilon_8 + 4\gamma g_1(m_2 + 2r_2), \quad \dot{\varepsilon}_9 = (-4+3\eta)\varepsilon_9 + \delta g_1^2 + 2\gamma g_1(2m_3 + r_3),\\
&\dot{\varepsilon}_{10} = (-4+3\eta)\varepsilon_{10} - 2\delta g_1 g_2 - 4\gamma g_2 m_3, \quad \dot{\varepsilon}_{11} = (-4+3\eta)\varepsilon_{11} - 4\gamma g_2 m_2, \quad \dot{\varepsilon}_{12} = (-4+3\eta)\varepsilon_{12},\\
&\dot{\varepsilon}_{13} = (-4+3\eta)\varepsilon_{13} - 4\delta g_1 g_2 - 4\gamma(g_1 r_3 + g_2 m_3), \quad \dot{\varepsilon}_{14} = (-4+3\eta)\varepsilon_{14} - 4\gamma g_2 m_2,\\
&\dot{\varepsilon}_{15} = (-4+3\eta)\varepsilon_{15} - 4\gamma g_1 r_2, \quad \dot{\varepsilon}_{16} = (-4+3\eta)\varepsilon_{16}, \quad \dot{\varepsilon}_{17} = (-4+3\eta)\varepsilon_{17} + 4\gamma g_1 m_2,\\
&\dot{\varepsilon}_{18} = (-4+3\eta)\varepsilon_{18} + 2\delta g_1^2 + 4\gamma g_1 m_3, \quad \dot{\varsigma}_1 = (-4+3\eta)\varsigma_1 + 2\gamma(g_1(4s_1 + s_2) + 3g_2 t), \quad \dot{\varsigma}_2 = (-4+3\eta)\varsigma_2,\\
&\dot{\varsigma}_3 = (-4+3\eta)\varsigma_3, \quad \dot{\varsigma}_4 = (-4+3\eta)\varsigma_4 - 4\gamma g_1 s_4, \quad \dot{\varsigma}_5 = (-4+3\eta)\varsigma_5 + 2\gamma g_1 s_2, \quad \dot{\varsigma}_6 = (-4+3\eta)\varsigma_6,\\
&\dot{\varsigma}_7 = (-4+3\eta)\varsigma_7 + 4\gamma g_1 s_4, \quad \dot{\varsigma}_8 = (-4+3\eta)\varsigma_8 + 2\gamma(g_1 s_2 - g_2 t), \quad \dot{\varsigma}_9 = (-4+3\eta)\varsigma_9 - 2\gamma g_2 t,\\
&\dot{\varsigma}_{10} = (-4+3\eta)\varsigma_{10} + 8\gamma g_1 s_4, \quad \dot{\varsigma}_{11} = (-4+3\eta)\varsigma_{11} + 2\gamma g_1(2s_3 + t), \quad \dot{\varsigma}_{12} = (-4+3\eta)\varsigma_{12} - 4\gamma g_1 s_3,\\
&\dot{\varsigma}_{13} = (-4+3\eta)\varsigma_{13} + 2\gamma g_1(-s_2 + t).
\end{aligned}
\tag{C.1}
$$

The only manifestation of the cut-off function comes through the constants α, β, γ and δ. However, as it happens in the bosonic case [19] we can reduce the number of independent parameters from four to two. In fact, by performing the following rescalings

$$
g^{(4,0)} \to \frac{1}{\alpha\gamma} g^{(4,0)}, \quad g^{(4,2)} \to \frac{\delta}{\alpha\gamma^2} g^{(4,2)}, \quad g^{(6,1)} \to \frac{1}{\alpha^2\gamma} g^{(6,1)}, \quad g^{(6,3)} \to \frac{\delta}{\alpha^2\gamma^2} g^{(6,3)}, \tag{C.2}
$$

where by $g^{(m,n)}$ we denote the coupling constants corresponding to the operators with m fermions and n derivatives, it can be shown that the β-functions depend on the scheme only through the combinations $z = \frac{\delta}{\gamma^2}$ and $w = \frac{\beta\delta}{\alpha\gamma}$. Moreover, z enters the equations only as a global factor of the anomalous dimension.

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Figure Captions

Fig. 1. $N\eta$ (solid line) and λ_1 (dashed line) as functions of N. This solution matches with the Type I solution of the large N limit.

Fig. 2. λ_1 as a function of w, ($z = 0.5$), for $N = 3$ and $N = 10$. The minimum clearly decreases with N.

Fig. 3. η (solid line) and λ_1 (dashed line) as a function of N for $z = 0.5$ and $w = -2$. In both curves the upper branch corresponds to the solution that matches with the Type II large N solution.

Fig. 4. η (solid line) and λ_1 (dashed line) as a function of N, ($z = 0.5$, $w = -2$) for a different fixed point solution. In this case both exponents are of order 1 as $N \to \infty$.

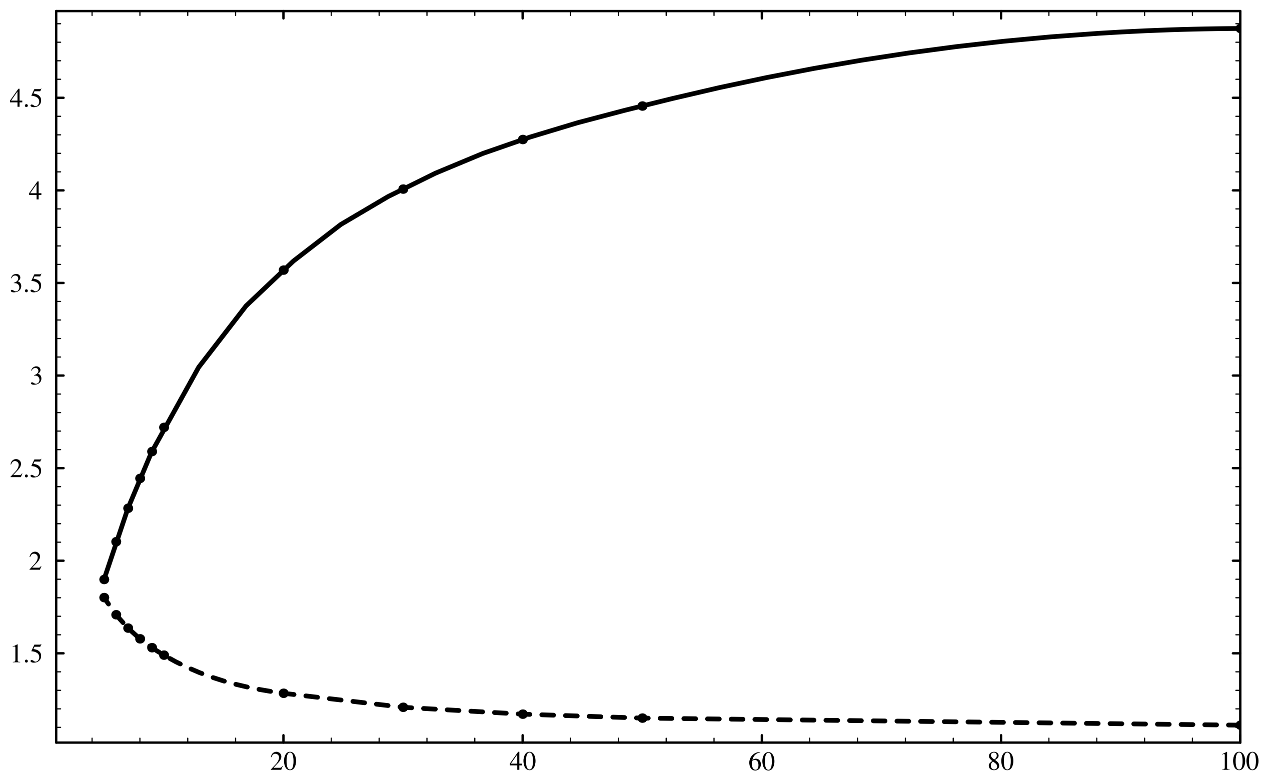

Fig. 1

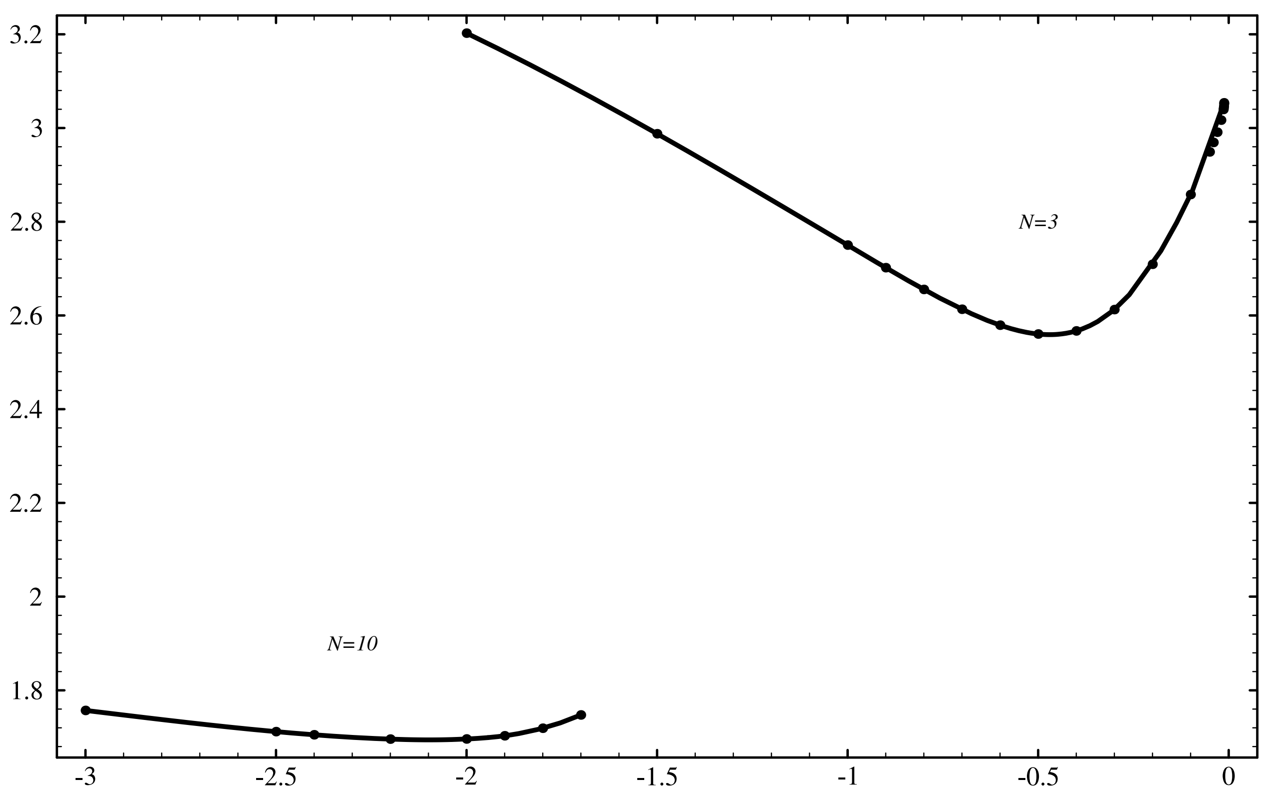

Fig. 2

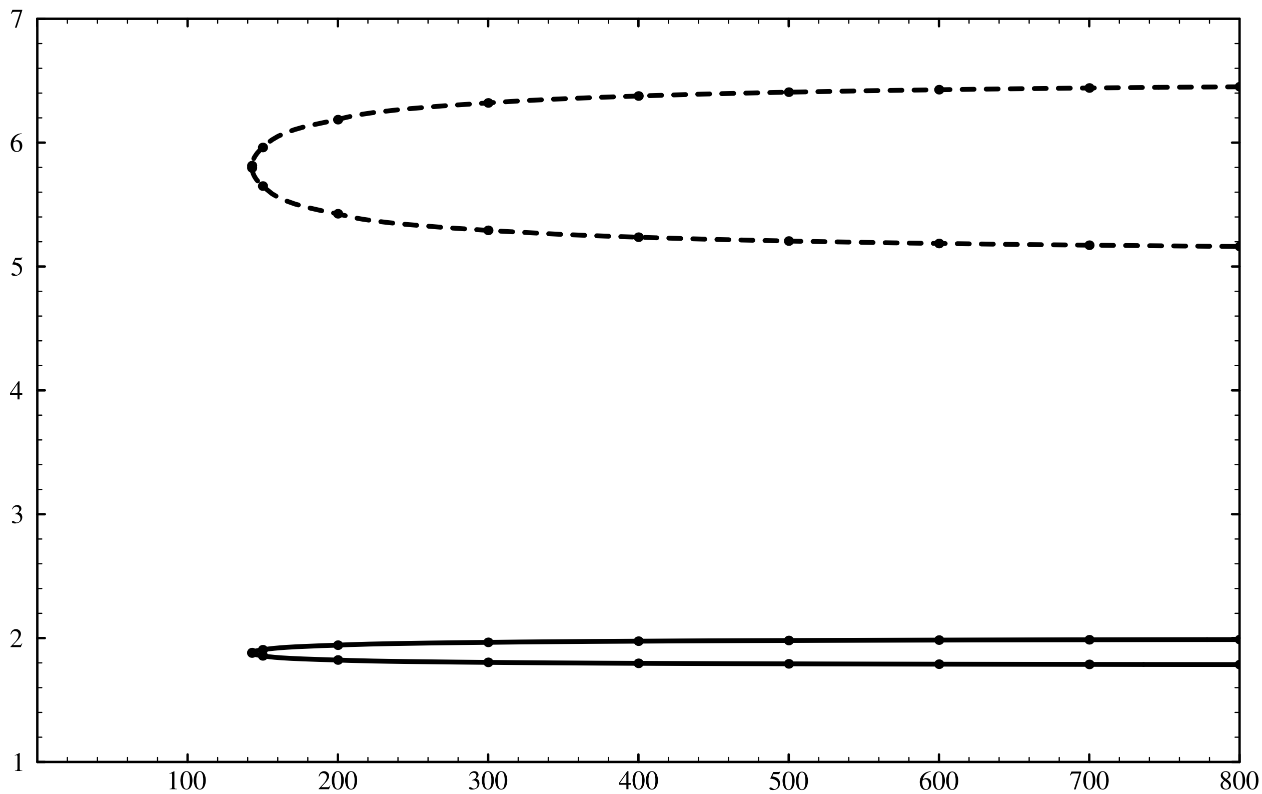

Fig. 3

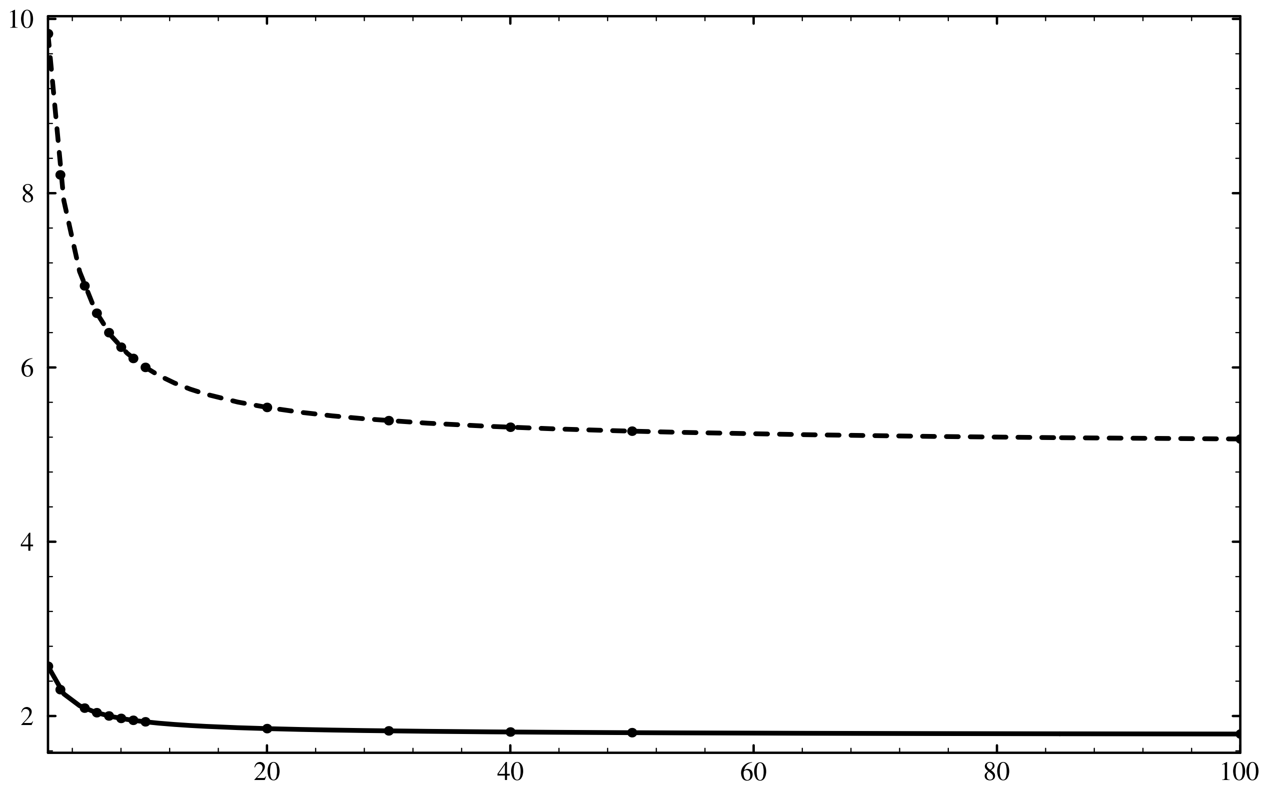

Fig. 4